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Algebraic Geometry over Free Groups: Lifting Solutions into Generic Points

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Abstract. In this paper we prove Implicit Function Theorems (IFT) for algebraic varieties defined by regular quadratic equations and, more generally, regular NTQ systems over free groups. In the model theoretic language these results state the existence of very simple Skolem functions for particular $\forall\exists$-formulas over free groups. We construct these functions effectively. In non-effective form IFT first appeared in [**18**]. From algebraic geometry view-point IFT can be described as lifting solutions of equations into generic points of algebraic varieties.

Moreover, we show that the converse is also true, i.e., IFT holds only for algebraic varieties defined by regular NTQ systems. This implies that if a finitely generated group H is $\forall\exists$-equivalent to a free non-abelian group then H is isomorphic to the coordinate group of a regular NTQ system.

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Introduction

The classical algebraic geometry is one of the main tools to deal with polynomial equations over fields. To study solutions of equations in free groups one needs a similar theory over groups. Recently basics of algebraic geometry over groups were developed in a series of papers [**2, 12, 13**]. This provides the necessary topological machinery to transcribe geometric notions into the language of pure group theory. In this paper, following [**2**] and [**12**], we freely use the standard algebraic geometric notions such as algebraic sets, the Zariski topology, Noetherian domains, irreducible varieties, radicals and coordinate groups to organize an approach to finding a solution of Tarski's problems in [**17**]. Our goal here is to prove several variations of so-called implicit function theorem (IFT) for free groups. The basic version of IFT was announced at the Model Theory conference at MSRI in 1998 [**14, 21**]. In [**15**] we used the basic version of implicit function theorem to solve the genus problem for quadratic non-orientable equations, and showed also that the abelianization of the cartesian power of infinitely many copies of a free non-abelian group has 2-torsion. The preprint [**18**] contains proofs of several variations of IFT in terms of liftings.

In a sense some formulations of IFT can be viewed as analogs of the corresponding results from analysis, hence the name. To demonstrate this we start with a very basic version of the implicit function theorem which holds for regular quadratic equations.

Let G be a group generated by A, $F(X)$ be a free group with basis $X = \{x_1, x_2, \ldots, x_n\}$, $G[X] = G * F(X)$ be a free product of G and $F(X)$. If $S \subset G[X]$ then the expression $S = 1$ is called *a system of equations* over G. A solution of the system $S = 1$ over G can be described as a G-homomorphism $\phi : G[X] \longrightarrow G$ such that $\phi(S) = 1$. By $V_G(S)$ we denote the set of all solutions in G of the system $S = 1$, it is called the *algebraic set defined by* S. This algebraic set $V_G(S)$ uniquely corresponds to the radical $R(S)$:

$$R(S) = \{T(x) \in G[X] \mid \forall A \in G^n (S(A) = 1 \to T(A) = 1\}.$$

The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the *coordinate group* of the algebraic set $V(S)$. Every solution of $S(X) = 1$ in G can be described as a G-homomorphism $G_{R(S)} \to G$.

Recall that a standard quadratic equation $S(X) = 1$ over group G is an equation in one of the following forms (below d, c_i are nontrivial elements from G):

$$\prod_{i=1}^{n}[x_i, y_i] = 1, \quad n > 0; \tag{1}$$

$$\prod_{i=1}^{n}[x_i, y_i] \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n, m \geqslant 0, m + n \geqslant 1; \tag{2}$$

$$\prod_{i=1}^{n} x_i^2 = 1, \quad n > 0; \tag{3}$$

$$\prod_{i=1}^{n} x_i^2 \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n, m \geqslant 0, n+m \geqslant 1. \tag{4}$$

Equations (1), (2) are called *orientable* and equations (3), (4) are called *non-orientable*. The numbers n and $n+m$ are called *genus* and *atomic rank* of $S(X) = 1$. Put

$$\kappa(S) = |X| + \varepsilon(S),$$

where $\varepsilon(S) = 1$ if the coefficient d occurs in S, and $\varepsilon(S) = 0$ otherwise. A standard quadratic equation $S(X) = 1$ is *regular* if $\kappa(S) \geqslant 4$ and there is a non-commutative solution of $S(X) = 1$ in G (see [**16**] for details), or it is an equation of the type $[x, y]d = 1$. Notice, that if $S(X) = 1$ has a solution in G, $\kappa(S) \geqslant 4$, and $n > 0$ in the orientable case ($n > 1$ in the non-orientable case), then the equation $S = 1$ has a non-commutative solution, hence regular.

Basic Form of IFT. *Let $S(X) = 1$ be a regular standard quadratic equation over a non-abelian free group F and let $T(X, Y) = 1$ be an equation over F, $|X| = m$, $|Y| = n$. Suppose that for any solution $U \in V_F(S)$ there exists a tuple of elements $W \in F^n$ such that $T(U, W) = 1$. Then there exists a tuple of words $P = (p_1(X), \ldots, p_n(X))$, with constants from F, such that $T(U, P(U)) = 1$ for any $U \in V_F(S)$. Moreover, one can fund a tuple P as above effectively.*

We define a Zariski topology on G^n by taking algebraic sets in G^n as a sub-basis for the closed sets of this topology. If G is a non-abelian fully residually free group (for every finite set of non-trivial elements in G there exists a homomorphism from G to a free group such that the images of these elements are non-trivial), then the closed sets in the Zariski topology over G are precisely the algebraic sets.

The Basic Form of IFT implies that locally (in terms of Zariski topology in F^n), i.e., in the neighborhood defined by the equation $S(X) = 1$, the implicit functions $y_1, \ldots, y_m$ can be expressed as explicit words in variables $x_1, \ldots, x_n$ and constants from F, say $Y = P(X)$. This allows one to eliminate a quantifier from the following formula (if it holds in a free group F)

$$\Phi = \forall X \exists Y (S(X) = 1 \quad \rightarrow \quad T(X, Y) = 1).$$

Indeed, in this event the sentence Φ is equivalent in F to the following one:

$$\Psi = \forall X (S(X) = 1 \quad \rightarrow \quad T(X, P(X)) = 1).$$

From the point of view of model theory Theorem A states the existence of very simple Skolem functions for particular $\forall\exists$-formulas over free groups. Observe, that Theorem A reinforces the results of [**18**] by giving the corresponding explicit Skolem functions effectively.

From algebraic geometry view-point the implicit function theorem tells one that (in the notations above) $T(X, Y) = 1$ has a solution at a generic point of the equation $S(X) = 1$. Indeed, since the coordinate group $F_{R(S)}$ of the equation $S(X) = 1$ is discriminated by the free group F the equation $T(X, Y) = 1$ has a solution in the group $F_{R(S)}$ (where elements from X are viewed as constants). This shows the Theorem A can be stated in the following form.

Theorem A'**.** *Let $S(X) = 1$ be a regular standard quadratic equation over a non-abelian free group F and let $T(X, Y) = 1$ be an equation over F, $|X| = m$, $|Y| = n$. Suppose that for any solution $U \in V_F(S)$ there exists a tuple of*

elements $W \in F^n$ *such that* $T(U,W) = 1$. *Then the equation* $T(X,Y) = 1$ *has a solution in the group* $F_{R(S)}$ (*where elements from* X *are viewed as constants from* $F_{R(S)}$).

This approach allows one to generalize the results above by replacing the equation $T(X,Y) = 1$ by an arbitrary system of equations and inequalities or even by an arbitrary boolean formula. Notice, that such generalizations in the form of Theorem A are impossible. To this end we need to introduce a few definitions.

Let $S(X) = 1$ be a system of equations over a group G which has a solution in G. We say that a system of equations $T(X,Y) = 1$ is *compatible* with $S(X) = 1$ over G if for every solution U of $S(X) = 1$ in G the equation $T(U,Y) = 1$ also has a solution in G. More generally, a formula $\Phi(X,Y)$ in the language L_A is *compatible* with $S(X) = 1$ over G, if for every solution $\bar{a}$ of $S(X) = 1$ in G there exists a tuple $\bar{b}$ over G such that the formula $\Phi(\bar{a},\bar{b})$ is true in G, i.e., the algebraic set $V_G(S)$ is a projection of the truth set of the formula $\Phi(X,Y) \wedge (S(X) = 1)$.

Suppose now that a formula $\Phi(X,Y)$ is compatible with $S(X) = 1$ over G. We say that $\Phi(X,Y)$ *admits a lift to a generic point* of $S = 1$ over G (or shortly S*-lift* over G), if the formula $\exists Y \Phi(X^\mu, Y)$ is true in $G_{R(S)}$ (here Y are variables and X^μ are constants from $G_{R(S)}$). Finally, an equation $T(X,Y) = 1$, which is compatible with $S(X) = 1$, admits a *complete* S*-lift* if every formula $T(X,Y) = 1 \;\&\; W(X,Y) \neq 1$, which is compatible with $S(X) = 1$ over G, admits an S-lift. We say that the lift (complete lift) is *effective* if there is an algorithm to decide for any equation $T(X,Y) = 1$ (any formula $T(X,Y) = 1 \;\&\; W(X,Y) \neq 1$) whether $T(X,Y) = 1$ (the formula $T(X,Y) = 1 \;\&\; W(X,Y) \neq 1$) admits an S-lift, and if it does, to construct a solution in $G_{R(S)}$.

Now the Implicit Function Theorem (IFT) for regular quadratic equations can be stated in the following general form. This is the main technical result of the paper, we prove it in Sections 3–6.

Theorem A. *Let* $S(X,A) = 1$ *be a regular standard quadratic equation over* $F(A)$. *Every equation* $T(X,Y,A) = 1$ *compatible with* $S(X,A) = 1$ *admits an effective complete* S*-lift.*

Furthermore, the IFT still holds if one replaces $S(X) = 1$ by an arbitrary system of a certain type, namely, by a *regular NTQ* system (see [**16**] for details). To explain this we need to introduce a few definitions.

Let G be a group with a generating set A. A system of equations $S = 1$ is called *triangular quasi-quadratic* (shortly, TQ) if it can be partitioned into the following subsystems

$$\begin{array}{rcl} S_1(X_1, X_2, \ldots, X_n, A) & = & 1 \\ S_2(X_2, \ldots, X_n, A) & = & 1 \\ & \vdots & \\ S_n(X_n, A) & = & 1 \end{array}$$

where for each i one of the following holds:

1) S_i is quadratic in variables X_i;
2) $S_i = \{[y,z] = 1, [y,u] = 1 \mid y,z \in X_i\}$ where u is a group word in $X_{i+1} \cup \cdots \cup X_n \cup A$ such that its canonical image in G_{i+1} is not a proper power. In this case we say that $S_i = 1$ corresponds to an extension of a centralizer;

3) $S_i = \{ [y,z] = 1 \mid y, z \in X_i \}$;
4) S_i is the empty equation.

Define $G_i = G_{R(S_i,\ldots,S_n)}$ for $i = 1, \ldots, n$ and put $G_{n+1} = G$. The TQ system $S = 1$ is called *non-degenerate* (shortly, NTQ) if each system $S_i = 1$, where $X_{i+1}, \ldots, X_n$ are viewed as the corresponding constants from G_{i+1} (under the canonical maps $X_j \to G_{i+1}$, $j = i+1, \ldots, n$, has a solution in G_{i+1}. The coordinate group of an NTQ system is called an *NTQ group*.

An NTQ system $S = 1$ is called *regular* if for each i the system $S_i = 1$ is either of the type 1) or 4), and in the former case the quadratic equation S_i is in standard form and regular.

In Section 8 we prove IFT for *regular NTQ* systems.

Theorem B. *Let $U(X, A) = 1$ be a regular NTQ-system. Every equation $V(X, Y, A) = 1$ compatible with $U = 1$ admits a complete effective U-lift.*

Notice, that by definition we allow empty equations in regular NTQ systems. In the case when the whole system $U = 1$ is empty there exists a very strong generalization of the basic implicit function theorem due to Merzljakov [**20**].

Merzljakov's Theorem. *If*

$$F \models \forall X_1 \exists Y_1 \cdots \forall X_k \exists Y_k (S(X, Y, A) = 1),$$

where $X = X_1 \cup \cdots \cup X_k, Y = Y_1 \cup \cdots \cup Y_k$, then there exist words (with constants from F) $q_1(X_1), \ldots, q_k(X_1, \ldots, X_k) \in F[X]$, such that

$$F[X] \models S(X_1, q_1(X_1), \ldots, X_k, q_k(X_1, \ldots, X_k, A)) = 1,$$

i.e., the equation

$$S(X_1, Y_1, \ldots, X_k, Y_k, A) = 1$$

(in variables Y) has a solution $Y_i = q_i(X_1, \ldots, X_i, A)$ in the free group $F[X]$, or equivalently,

$$F \models \forall X_1 \ldots \forall X_n (S(X_1, q_1(X_1, A), \ldots, X_k, q_k(X_1, \ldots, X_k, A)) = 1).$$

In [**18**] we gave a short proof of Merzljakov's theorem based on generalized equations. The key idea of all known proofs of this result is to consider a set of *Merzljakov's words* as values of variables from $X_i = \{x_{i1}, \ldots, x_{ik_i}\}$:

$$x_{ij} = ba^{m_{ij1}}ba^{m_{ij2}}b \cdots ba^{m_{ijn_{ij}}}b,$$

where a, b are two different generators of $F = F(A)$. If $S(X, Y, A) = 1$ has a solution for any Merzljakov' words as values of variables from X, then it has a solution of the type $Y_i = q_i(X_1, \ldots, X_i)$, $i = 1, \ldots, k$.

Unfortunately, Merzljakov's words are not, in general, solutions of a regular quadratic equation $S(X) = 1$ over F. In this case, one needs to find sufficiently many solutions of $S(X) = 1$ over F with sufficiently complex periodic structure of subwords. To this end we consider analogs of Merzljakov's words in the group of automorphisms of $F[X]$ that fix the standard quadratic word $S(X)$ and the corresponding set of solutions of $S(X) = 1$ in F. In Sections 4 and 5 we study in detail the periodic structure of these solutions. This is the most technically demanding part of the paper.

There are two more important generalizations of the implicit function theorem, one – for arbitrary NTQ-systems, and another – for arbitrary systems. We need a few more definitions to explain this. Let $U(X_1,\dots,X_n,A)=1$ be an NTQ-system:

$$\begin{aligned} S_1(X_1,X_2,\dots,X_n,A) &= 1\\ S_2(X_2,\dots,X_n,A) &= 1\\ &\vdots\\ S_n(X_n,A) &= 1\end{aligned}$$

and $G_i = G_{R(S_i,\dots,S_n)}$, $G_{n+1}=F(A)$.

A G_{i+1}-automorphism σ of G_i is called a *canonical automorphism* if the following holds:

1) if S_i is quadratic in variables X_i then σ is induced by a G_{i+1}-automorphism of the group $G_{i+1}[X_i]$ which fixes S_i;
2) if $S_i = \{[y,z]=1,[y,u]=1 \mid y,z\in X_i\}$ where u is a group word in $X_{i+1}\cup\cdots\cup X_n\cup A$, then $G_i = G_{i+1}*_{u=u}Ab(X_i\cup\{u\})$, where $Ab(X_i\cup\{u\})$ is a free abelian group with basis $X_i\cup\{u\}$, and in this event σ extends an automorphism of $Ab(X_i\cup\{u\})$ (which fixes u);
3) If $S_i=\{[y,z]=1\mid y,z\in X_i\}$ then $G_i=G_{i+1}*Ab(X_i)$, and in this event σ extends an automorphism of $Ab(X_i)$;
4) If S_i is the empty equation then $G_i=G_{i+1}[X_i]$, and in this case σ is just the identity automorphism of G_i.

Let π_i be a fixed $G_{i+1}[Y_i]$-homomorphism

$$\pi_i : G_i[Y_i]\to G_{i+1}[Y_{i+1}],$$

where $\emptyset = Y_1\subseteq Y_2\subseteq\ldots\subseteq Y_n\subseteq Y_{n+1}$ is an ascending chain of finite sets of parameters, and $G_{n+1}=F(A)$. Since the system $U=1$ is non-degenerate such homomorphisms π_i exist. We assume also that if $S_i(X_i)=1$ is a standard quadratic equation (the case 1) above) which has a non-commutative solution in G_{i+1}, then X^{π_i} is a non-commutative solution of $S_i(X_i)=1$ in $G_{i+1}[Y_{i+1}]$.

A *fundamental sequence* (or a *fundamental set*) of solutions of the system $U(X_1,\dots,X_n,A)=1$ in $F(A)$ with respect to the fixed homomorphisms $\pi_1,\dots,\pi_n$ is a set of all solutions of $U=1$ in $F(A)$ of the form

$$\sigma_1\pi_1\cdots\sigma_n\pi_n\tau,$$

where σ_i is Y_i-automorphism of $G_i[Y_i]$ induced by a canonical automorphism of G_i, and τ is an $F(A)$-homomorphism $\tau : F(A\cup Y_{n+1})\to F(A)$. Solutions from a given fundamental set of U are called *fundamental* solutions.

Theorem C (Parametrization theorem). *Let $U(X,A)=1$ be an NTQ-system and $V_{\text{fund}}(U)$ a fundamental set of solutions of $U=1$ in $F=F(A)$. If a formula*

$$\Phi = \forall X(U(X)=1\to\exists Y(W(X,Y,A)=1\wedge W_1(X,Y,A)\neq 1)$$

is true in F then one can effectively find finitely many NTQ systems $U_1=1,\dots,U_k=1$ and embeddings $\theta_i : F_{R(U)}\to F_{R(U_i)}$ such that the formula

$$\exists Y(W(X^{\theta_i},Y,A)=1\wedge W_1(X^{\theta_i},Y,A)\neq 1)$$

is true in each group $F_{R(U_i)}$. Furthermore, for every solution $\phi : F_{R(U)} \to F$ of $U = 1$ from $V_{\text{fund}}(U)$ there exists $i \in \{1, \ldots, k\}$ and a fundamental solution $\psi : F_{R(U_i)} \to F$ such that $\phi = \theta_i \psi$.

As a corollary of this theorem and results from [**16**, Section 11], we obtain the following result.

Theorem D. *Let $S(X) = 1$ be an arbitrary system of equations over F. If a formula*

$$\Phi = \forall X \exists Y (S(X) = 1 \quad \to \quad (W(X, Y, A) = 1 \wedge W_1(X, Y, A) \neq 1))$$

is true in F then one can effectively find finitely many NTQ systems $U_1 = 1, \ldots, U_k = 1$ and F-homomorphisms $\theta_i : F_{R(S)} \to F_{R(U_i)}$ such that the formula

$$\exists Y (W(X^{\theta_i}, Y, A) = 1 \wedge W_1(X^{\theta_i}, Y, A) \neq 1)$$

is true in each group $F_{R(U_i)}$. Furthermore, for every solution $\phi : F_{R(S)} \to F$ of $S = 1$ there exists $i \in \{1, \ldots, k\}$ and a fundamental solution $\psi : F_{R(U_i)} \to F$ such that $\phi = \theta_i \psi$.

In Section 9 we show that the converse of Theorem B holds. Namely, we prove the following theorem.

Theorem E. *Let F be a free non-abelian group and $S(X) = 1$ a consistent system of equations over F. Then the following conditions are equivalent:*

(1) *The system $S(X) = 1$ is rationally equivalent to a regular NTQ system.*
(2) *Every equation $T(X, Y) = 1$ which is compatible with $S(X) = 1$ over F admits an S-lift.*
(3) *Every equation $T(X, Y) = 1$ which is compatible with $S(X) = 1$ over F admits a complete S-lift.*

Theorem E immediately implies the following remarkable property of regular NTQ systems. Denote by L_A the first-order group theory language with constants from the free group $F(A)$. If Φ is a set of first order sentences of the language L_A then two groups G and H are called Φ-*equivalent* if they satisfy precisely the same sentences from the set Φ. In this event we write $G \equiv_\Phi H$. In particular, $G \equiv_{\forall\exists} H$ ($G \equiv_{\exists\forall} H$) means that G and H satisfy precisely the same $\forall\exists$-sentences ($\exists\forall$-sentences). We have shown in [**13**] that for a finitely generated group G if $G \equiv_{\forall\exists} H$ then G is torsion-free hyperbolic and fully residually free. Now we improve on this result.

Theorem F. *Let G be a finitely generated group. If G is $\forall\exists$-equivalent to a free non-abelian group F then G is isomorphic to the coordinate group $F_{R(S)}$ of a regular NTQ system $S = 1$ over F.*

Notice, that we prove in the consequent paper [**17**] that the converse is also true, moreover, it holds in the strongest possible form. Namely, the coordinate group $F_{R(S)}$ of a regular NTQ system $S = 1$ over F is elementary equivalent to a free non-abelian group F. Combining this result with Theorem E one obtains a complete algebraic characterization of finitely generated groups which are elementary equivalent to a free non-abelian group. Similar characterization in different terms is given in [**26**].

1. Scheme of the proof

We sketch here the proof of Theorem A for the orientable quadratic equation.

$$\prod_{i=1}^{n}[x_i, y_i]\prod_{i=1}^{m} z_i^{-1}c_iz_ic = 1, \quad n \geqslant 1, m+n \geqslant 1, c \neq 1. \tag{5}$$

We begin with the definition of compatibility. Let X, Y be families of variables

DEFINITION 1.1. Let $S(X) = 1$ be a system of equations over a group G which has a solution in G. We say that a system of equations $T(X, U) = 1$ is compatible with $S(X) = 1$ over G if for every solution B of $S(X) = 1$ in G the equation $T(B, U) = 1$ also has a solution in G.

Let $F = F(A)$ be a free group with alphabet A. Denote by $S(X) = 1$ equation (5), where $X = \{x_1, y_1, \dots, x_n, y_n, z_1, \dots, z_m\}$, and suppose that an equation $T(X, U)1$ is compatible with $S(X) = 1$.

STEP 1. The following statement can be obtained using the Elimination process similar to Makanin-Razborov's process described in [**16**].

One can effectively find a finite disjunction of systems $\Pi(M, X)$ of graphic equations (without cancellation) in variables M, X with the following properties.

1) Each equation in the system $\Pi(M, X)$ has form $x \doteq \mu_{i_1} \circ \cdots \circ \mu_{i_k}$, where $x \in X$, $\mu_i \in M$, "$\doteq$" stands for graphic equality and "$\circ$" means multiplication without cancellation. A solution of such a graphic equation is a tuple of reduced words $x^\alpha, \mu_{i_1}^\alpha, \dots, \mu_{i_k}^\alpha$ in F such that x^α is graphically equal to the product $\mu_{i_1}^\alpha \circ \cdots \circ \mu_{i_k}^\alpha$.
2) For every solution B of $S(X) = 1$ written in reduced form there exists a graphic solution B, D of one of the systems $\Pi(M, X)$ in this disjunction.
3) Let $U = \{u_1, \dots, u_k\}$. For every system $Q(X, M)$ one can find words $f_1(M), \dots, f_k(M)$ such that for every solution B, D (not necessary graphic) of the system $Q(X, M)$ in F one has $T(X, f_1(D), \dots, f_k(D)) = 1$.

Such system of graphic equations $\Pi(M, X)$ is called in Section 3 a "cut equation" (see Definition 3.1 and Theorem 3.4.) Indeed, variables X are "cut" into pieces. We can think about the cut equation as a system of intervals labelled by solutions of $S(X) = 1$ that are cut into several parts corresponding to variables in M.

STEP 2. Now we construct a discriminating family of solutions of $S(X) = 1$ (see the definition in [**16**, Section 1.4]) which later will be called a *generic* family. Consider a group $F[X] = F * F(X)$ and construct a particular sequence $\Gamma = (\gamma_1, \dots, \gamma_K)$ of F-automorphisms of $F[X]$ that fix the quadratic word $S(X)$. This is done in Section 4. These automorphisms have the property that any two neighbors in the sequence do not commute and it is in some sense maximal with this property. For any natural number j define $\gamma_j = \gamma_r$, where r is the remainder when j is divided by K.

For example, for the equation $[x, y] = [a, b]$ we can take

$$\gamma_1 : x \to x, \ y \to xy;$$

$$\gamma_2 : x \to yx, \ y \to y,$$

in this case $K = 2$ and

$$\gamma_{2s-1} = \gamma_1, \gamma_{2s} = \gamma_2.$$

These automorphisms are, actually, Dehn twists. Notice that

$$\gamma_1^q : x \to x,\ y \to x^q y;\quad \gamma_2^q : x \to y^q x,\ y \to y,$$

therefore big powers of automorphisms produce big powers of elements. Let L be a multiple of K. Define

$$\phi_{L,p} = \gamma_L^{p_L} \gamma_{L-1}^{p_{L-1}} \cdots \gamma_1^{p_1},$$

where $p = (p_1, \ldots, p_L)$. Now we take a suitable (with small cancellation, in general position) solution of $S(X) = 1$. Denote $F_{\mathrm{Rad}(S)} = F * F[X]/\mathrm{ncl}(S)$. This solution is a homomorphism $\beta : F_{\mathrm{Rad}(S)} \to F$. The family of mappings

$$\Psi_L = \{\psi_{L,p} = \phi_{L,p}\beta,\ p \in P\},$$

where L is large and P is an infinite set of L-tuples of large natural numbers, is a family of solutions of $S(X) = 1$. It is very important that this is a discriminating family.

For example, take for the equation $[x, y] = [a, b]$ $x^\beta = a, y^\beta = b$, then for $L = 4$ we have

$$x = (((a^{p_1}b)^{p_2}a)^{p_3}a^{p_1}b)^{p_4}(a^{p_1}b)^{p_2}a, \quad y = ((a^{p_1}b)^{p_2}a)^{p_3}a^{p_1}b. \tag{6}$$

The word $((a^{p_1}b)^{p_2}a)^{p_3}a^{p_1}b$ is called a period in rank 4. Notice that the period of rank 4 is, actually, $y^{\psi_{3,p}}$.

Since the family of cut equations is finite, some infinite set of solutions X^{Ψ_L} satisfies the same cut equation $\Pi(M, X)$. Therefore, it is enough to consider one of the cut equations $\Pi(M, X)$.

In the example (6) there is no cancellation between a and b and, therefore, it does not matter whether we label intervals of the cut equation by $X^{\psi_{L,p}}$ or by $X^{\phi_{L,p}}$. In Section 5 we show how to choose a solution β with relatively small cancellation, so that we can forget about the cancellation and label the intervals of $\Pi(M, X)$ by $X^{\phi_{L,p}}$.

STEP 3. We can see now that for different L-tuples p all values of $X^{\phi_{L,p}}$ (in $F[X]$) have similar periodic structure and must be "cut " the same way into pieces $\mu \in M$. Therefore big powers are similarly distributed between pieces $\mu \in M$. In Section 7 we introduce the notion of *complexity* of a cut equation.

Let $\Pi(M, X)$ be a cut equation. For a positive integer n by $k_n(\Pi)$ we denote the number of equations (intervals) $x \doteq \mu_{i_1} \circ \cdots \circ \mu_{i_n}$ that have right hand side of length n. The following sequence of integers

$$Comp(\Pi) = \big(k_2(\Pi), k_3(\Pi), \ldots, k_{\mathrm{length}(\Pi)}(\Pi)\big)$$

is called the *complexity* of Π.

We well-order complexities of cut equations in the (right) shortlex order: if Π and Π' are two cut equations then $\mathrm{Comp}(\Pi) < \mathrm{Comp}(\Pi')$ if and only if $\mathrm{length}(\Pi) < \mathrm{length}(\Pi')$ or $\mathrm{length}(\Pi) = \mathrm{length}(\Pi')$ and there exists $1 \leqslant i \leqslant \mathrm{length}(\Pi)$ such that $k_j(\Pi) = k_j(\Pi')$ for all $j > i$ but $k_i(\Pi) < k_i(\Pi')$.

Observe that equations of the form $x \doteq \mu_i$ have no input into the complexity of a cut equation. In particular, cut equations that have all graphic equations of length one have the minimal possible complexity among equations of a given length. We will write $\mathrm{Comp}(\Pi) = 0$ in the case when $k_i(\Pi)0$ for every $i = 2, \ldots, \mathrm{length}(\Pi)$.

We introduce the process of transformations of the cut equation $\Pi(M, X)$. This process consists in "cutting out" big powers of largest periods from the interval and replacing one interval labelled by $X^{\phi_{i,p}}$ by several intervals labelled by

$X^{\phi_{i-1,p}}$. After such a transformation the left sides of the graphic equalities in the cut equation correspond to values $X^{\phi_{i-1,p}}$ (or very short words in $X^{\phi_{i-1,p}}$) and the complexity either decreases or stabilizes during several steps of the process. Suppose $\mathrm{Comp}(\Pi) = 0$ after t transformations, so that each graphic equality has form $x^{\phi_{L-t,p}} \doteq \mu$ or $y^{\phi_{L-t,p}} \doteq \nu$. Therefore, $x^{\psi_{L-t,p}} \doteq \mu$ or $y^{\psi_{L-t,p}} \doteq \nu$ for a discriminating family of solutions $\Psi_{L-t,P}$. By the properties of discriminating families, $\mu = x$, $\nu = y$ in the group $F_{\mathrm{Rad}(S)}$. Substituting μ and ν into words $f_1, \ldots, f_k$ we obtain a solution U of the equation $T(X, U) = 1$ given by a formula in x, y in $F_{\mathrm{Rad}(S)}$.

In a general case, when the length of the right hand side of the cut equation does not decrease during several steps in the process of transformations, the situation is, certainly, a bit more complicated. In this case one can show that in each graphic equation all the variables μ_i except the first and the last one are very short and can be taken almost arbitrary, and the other variables can be expressed in terms of them and $X^{\Psi_{L-t,P}}$.

2. Elementary properties of liftings

In this section we discuss some basic properties of liftings of equations and inequalities into generic points.

Let G be a group and let $S(X) = 1$ be a system of equations over G. Recall that by G_S we denote the quotient group $G[X]/\mathrm{ncl}(S)$, where $\mathrm{ncl}(S)$ is the normal closure of S in $G[X]$. In particular, $G_{R(S)} = G[X]/R(S)$ is the coordinate group defined by $S(X) = 1$. The radical $R(S)$ can be described as follows. Consider a set of G-homomorphisms

$$\Phi_{G,S} = \{\phi \in \mathrm{Hom}_G(G[S], G) \mid \phi(S) = 1\}.$$

Then

$$R(S) = \begin{cases} \bigcap_{\phi \in \Phi_{G,S}} \ker \phi & \text{if } \Phi_{G,S} \neq \emptyset \\ G[X] & \text{otherwise} \end{cases}$$

Now we put these definitions in a more general framework. Let H and K be G-groups and $M \subset H$. Put

$$\Phi_{K,M} = \{\phi \in \mathrm{Hom}_G(H, K) \mid \phi(M) = 1\}.$$

Then the following subgroup is termed the *G-radical of M with respect to K:*

$$\mathrm{Rad}_K(M) = \begin{cases} \bigcap_{\phi \in \Phi_{K,M}} \ker \phi, & \text{if } \Phi_{K,M} \neq \emptyset, \\ G[X] & \text{otherwise.} \end{cases}$$

Sometimes, to emphasize that M is a subset of H, we write $\mathrm{Rad}_K(M, H)$. Clearly, if $K = G$, then $R(S) = \mathrm{Rad}_G(S, G[X])$.

Let

$$H_K^* = H/\mathrm{Rad}_K(1).$$

Then H_K^* is either a G-group or trivial. If $H_K^* \neq 1$, then it is G-separated by K. In the case $K = G$ we omit K in the notation above and simply write H^*. Notice that

$$(H/\mathrm{ncl}(M))_K^* \simeq H/\mathrm{Rad}_K(M),$$

in particular, $(G_S)^* G_{R(S)}$.

LEMMA 2.1. *Let $\alpha : H_1 \to H_2$ be a G-homomorphism and suppose*

$$\Phi = \{\phi : H_2 \to K\}$$

be a separating family of G-homomorphisms. Then

$$\ker \alpha = \bigcap \{\ker(\alpha\phi) \ \mid \ \phi \in \Phi\}$$

PROOF. Suppose $h \in H_1$ and $h \notin \ker(\alpha)$. Then $\alpha(h) \neq 1$ in H_2. Hence there exists $\phi \in \Phi$ such that $\phi(\alpha(h)) \neq 1$. This shows that $\ker \alpha \supset \bigcap\{\ker(\alpha \circ \phi) \ \mid \ \phi \in \Phi\}$. The other inclusion is obvious. □

LEMMA 2.2. *Let H_1, H_2, and K be G-groups.*

(1) *Let $\alpha : H_1 \to H_2$ be a G-homomorphism and let H_2 be G-separated by K. If $M \subset \ker \alpha$, then $\mathrm{Rad}_K(M) \subseteq \ker \alpha$.*
(2) *Every G-homomorphism $\phi : H_1 \to H_2$ gives rise to a unique homomorphism*

$$\phi^* : (H_1)^*_K \to (H_2)^*_K$$

such that $\phi\eta_2 = \eta_1\phi^*$, where $\eta_i : H_i \to H_i^*$ is the canonical epimorphism.

PROOF. (1) We have

$$\begin{aligned} \mathrm{Rad}_K(M, H_1) &= \bigcap\{\ker \phi \mid \phi : H_1 \to_G K \ \wedge \ \phi(M) = 1\} \\ &\subseteq \bigcap\{\ker(\alpha\beta) \mid \beta : H_2 \to_G K\} \\ &= \ker \alpha. \end{aligned}$$

(2) Let $\alpha : H_1 \to (H_2)^*_K$ be the composition of the following homomorphisms

$$H_1 \xrightarrow{\phi} H_2 \xrightarrow{\eta_2} (H_2)^*_K.$$

Then by assertion 1 $\mathrm{Rad}_K(1, H_1) \subseteq \ker \alpha$, therefore α induces the canonical G-homomorphism $\phi^* : (H_1)^*_K \to (H_2)^*_K$. □

LEMMA 2.3.

(1) *The canonical map $\lambda : G \to G_S$ is an embedding $\iff$ $S(X) = 1$ has a solution in some G-group H.*
(2) *The canonical map $\mu : G \to G_{R(S)}$ is an embedding $\iff$ $S(X) = 1$ has a solution in some G-group H which is G-separated by G.*

PROOF. (1) If $S(x_1, \ldots, x_m) = 1$ has a solution $(h_1, \ldots, h_m)$ in some G-group H, then the G-homomorphism $x_i \to h_i$, $(i = 1, \ldots, m)$ from $G[x_1, \ldots, x_m]$ into H induces a homomorphism $\phi : G_S \to H$. Since H is a G-group all non-trivial elements from G are also non-trivial in the factor-group G_S, therefore $\lambda : G \to G_S$ is an embedding. The converse is obvious.

(2) Let $S(x_1, \ldots, x_m) = 1$ have a solution $(h_1, \ldots, h_m)$ in some G-group H which is G-separated by G. Then there exists the canonical G-homomorphism $\alpha : G_S \to H$ defined as in the proof of the first assertion. Hence $R(S) \subseteq \ker \alpha$ by Lemma 2.2, and α induces a homomorphism from $G_{R(S)}$ into H, which is monic on G. Therefore G embeds into $G_{R(S)}$. The converse is obvious. □

Now we apply Lemma 2.2 to coordinate groups of nonempty algebraic sets.

Lemma 2.4. *Let subsets S and T from $G[X]$ define non-empty algebraic sets in a group G. Then every G-homomorphism $\phi : G_S \to G_T$ gives rise to a G-homomorphism $\phi^* : G_{R(S)} \to G_{R(T)}$.*

Proof. The result follows from Lemma 2.2 and Lemma 2.3. □

Now we are in a position to give the following

Recall that for a consistent system of equations $S(X) = 1$ over a group G, a system of equations $T(X,Y) = 1$ is compatible with $S(X) = 1$ over G if for every solution U of $S(X) = 1$ in G the equation $T(U,Y) = 1$ also has a solution in G, i.e., the algebraic set $V_G(S)$ is a projection of the algebraic set $V_G(S \cup T)$.

The next proposition describes compatibility of two equations in terms of their coordinate groups.

Proposition 2.5. *Let $S(X) = 1$ be a system of equations over a group G which has a solution in G. Then $T(X,Y) = 1$ is compatible with $S(X) = 1$ over G if and only if $G_{R(S)}$ is canonically embedded into $G_{R(S \cup T)}$, and every G-homomorphism $\alpha : G_{R(S)} \to G$ extends to a G-homomorphisms $\alpha' : G_{R(S \cup T)} \to G$.*

Proof. Suppose first that $T(X,Y) = 1$ is compatible with $S(X) = 1$ over G and suppose that $V_G(S) \neq \emptyset$. The identity map $X \to X$ gives rise to a G-homomorphism

$$\lambda : G_S \longrightarrow G_{S \cup T}$$

(notice that both G_S and $G_{S \cup T}$ are G-groups by Lemma 2.3), which by Lemma 2.4 induces a G-homomorphism

$$\lambda^* : G_{R(S)} \longrightarrow G_{R(S \cup T)}.$$

We claim that λ^* is an embedding. To show this we need to prove first the statement about the extensions of homomorphisms. Let $\alpha : G_{R(S)} \to G$ be an arbitrary G-homomorphism. It follows that $\alpha(X)$ is a solution of $S(X) = 1$ in G. Since $T(X,Y) = 1$ is compatible with $S(X) = 1$ over G, there exists a solution, say $\beta(Y)$, of $T(\alpha(X), Y) = 1$ in G. The map

$$X \to \alpha(X), Y \to \beta(Y)$$

gives rise to a G-homomorphism $G[X,Y] \to G$, which induces a G-homomorphism $\phi : G_{S \cup T} \to G$. By Lemma 2.4 ϕ induces a G-homomorphism

$$\phi^* : G_{R(S \cup T)} \longrightarrow G.$$

Clearly, ϕ^* makes the following diagram to commute.

$$\begin{tikzcd} G_{R(S)} \arrow[r, "\lambda^*"] \arrow[d, "\alpha"'] & G_{R(S \cup T)} \arrow[dl, "\phi^*"] \\ G & \end{tikzcd}$$

Now to prove that λ^* is an embedding, observe that $G_{R(S)}$ is G-separated by G. Therefore for every non-trivial $h \in G_{R(S)}$ there exists a G-homomorphism $\alpha : G_{R(S)} \to G$ such that $\alpha(h) \neq 1$. But then $\phi^*(\lambda^*(h)) \neq 1$ and consequently $h \notin \ker \lambda^*$. The converse statement is obvious. □

Let $S(X) = 1$ be a system of equations over G and suppose $V_G(S) \neq \emptyset$. The canonical embedding $X \to G[X]$ induces the canonical map

$$\mu : X \to G_{R(S)}.$$

We are ready to formulate the main definition.

Definition 2.6. Let $S(X) = 1$ be a system of equations over G with $V_G(S) \neq \emptyset$ and let $\mu : X \to G_{R(S)}$ be the canonical map. Let a system $T(X, Y) = 1$ be compatible with $S(X) = 1$ over G. We say that $T(X, Y) = 1$ admits a lift to a generic point of $S = 1$ over G (or, shortly, S-lift over G) if $T(X^\mu, Y) = 1$ has a solution in $G_{R(S)}$ (here Y are variables and X^μ are constants from $G_{R(S)}$).

Lemma 2.7. *Let $T(X, Y) = 1$ be compatible with $S(X) = 1$ over G. If $T(X, Y) = 1$ admits an S-lift, then the identity map $Y \to Y$ gives rise to a canonical $G_{R(S)}$-epimorphism from $G_{R(S\cup T)}$ onto the coordinate group of $T(X^\mu, Y) = 1$ over $G_{R(S)}$:*

$$\psi^* : G_{R(S\cup T)} \to G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y)).$$

Moreover, every solution U of $T(X^\mu, Y) = 1$ in $G_{R(S)}$ gives rise to a $G_{R(S)}$-homomorphism $\phi_U : G_{R(S\cup T)} \to G_{R(S)}$, where $\phi_U(Y) = U$.

Proof. Observe that the following chain of isomorphisms hold:

$$\begin{aligned} G_{R(S\cup T)} &\simeq_G G[X][Y]/\mathrm{Rad}_G(S \cup T) \\ &\simeq_G G[X][Y]/\mathrm{Rad}_G(\mathrm{Rad}_G(S, G[X]) \cup T) \\ &\simeq_G (G[X][Y]/\mathrm{ncl}(\mathrm{Rad}_G(S, G[X]) \cup T))^* \\ &\simeq_G \left(G_{R(S)}[Y]/\mathrm{ncl}(T(X^\mu, Y))\right)^*. \end{aligned}$$

Denote by $\overline{G_{R(S)}}$ the canonical image of $G_{R(S)}$ in $(G_{R(S)}[Y]/\mathrm{ncl}(T(X^\mu, Y)))^*$.

Since $\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y))$ is a normal subgroup in $G_{R(S)}[Y]$ containing $T(X^\mu, Y)$ there exists a canonical G-epimorphism

$$\psi : G_{R(S)}[Y]/\mathrm{ncl}(T(X^\mu, Y)) \to G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y)).$$

By Lemma 2.2 the homomorphism ψ gives rise to a canonical G-homomorphism

$$\psi^* : (G_{R(S)}[Y]/\mathrm{ncl}(T(X^\mu, Y)))^* \to (G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y)))^*.$$

Notice that the group $G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y))$ is the coordinate group of the system $T(X^\mu, Y) = 1$ over $G_{R(S)}$ and this system has a solution in $G_{R(S)}$. Therefore this group is a $G_{R(S)}$-group and it is $G_{R(S)}$-separated by $G_{R(S)}$. Now since $G_{R(S)}$ is the coordinate group of $S(X) = 1$ over G and this system has a solution in G, we see that $G_{R(S)}$ is G-separated by G. It follows that the group $G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y))$ is G-separated by G. Therefore

$$G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y)) = (G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y)))^*.$$

Now we can see that

$$\psi^* : G_{R(S\cup T)} \to G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y))$$

is a G-homomorphism which maps the subgroup $\overline{G_{R(S)}}$ from $G_{R(S\cup T)}$ onto the subgroup $G_{R(S)}$ in $G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y))$.

This shows that $\overline{G_{R(S)}} \simeq_G G_{R(S)}$ and ψ^* is a $G_{R(S)}$-homomorphism. If U is a solution of $T(X^\mu, Y) = 1$ in $G_{R(S)}$, then there exists a $G_{R(S)}$-homomorphism

$$\phi_U : G_{R(S)}[Y]/\mathrm{Rad}_{G_{R(S)}}(T(X^\mu, Y)) \to G_{R(S)}.$$

such that $\phi_U(Y) = U$. Obviously, composition of ϕ_U and ψ^* gives a $G_{R(S)}$-homomorphism from $G_{R(S\cup T)}$ into $G_{R(S)}$, as desired. $\square$

The next result characterizes lifts in terms of the coordinate groups of the corresponding equations.

PROPOSITION 2.8. *Let $S(X) = 1$ be an equation over G which has a solution in G. Then for an arbitrary equation $T(X, Y) = 1$ over G the following conditions are equivalent:*

(1) *$T(X, Y) = 1$ is compatible with $S(X) = 1$ and $T(X, Y) = 1$ admits S-lift over G;*
(2) *$G_{R(S)}$ is a retract of $G_{R(S,T)}$, i.e., $G_{R(S)}$ is a subgroup of $G_{R(S,T)}$ and there exists a $G_{R(S)}$-homomorphism $G_{R(S,T)} \to G_{R(S)}$.*

PROOF. (1) $\Longrightarrow$ (2). By Proposition 2.5 $G_{R(S)}$ is a subgroup of $G_{R(S,T)}$. Moreover, $T(X^\mu, Y) = 1$ has a solution in $G_{R(S)}$, so by Lemma 2.7 there exists a $G_{R(S)}$-homomorphism $G_{R(S,T)} \to G_{R(S)}$, i.e., $G_{R(S)}$ is a retract of $G_{R(S,T)}$.

(2) $\Longrightarrow$ (1). If $\phi : G_{R(S,T)} \to G_{R(S)}$ is a retract then every G-homomorphism $\alpha : G_{R(S)} \to G$ extends to a G-homomorphism $\phi\alpha : G_{R(S,T)} \to G$. It follows from Proposition 2.5 that $T(X, Y) = 1$ is compatible with $S(X) = 1$ and ϕ gives a solution of $T(X^\mu, Y) = 1$ in $G_{R(S)}$, as desired. $\square$

Denote by $\mathcal{C}$ (respectively $\mathcal{C}^*$) the class of all finite systems $S(X) = 1$ over F such that every equation $T(X, Y) = 1$ compatible with $S = 1$ admits an S-lift (complete S-lift).

The following result shows that the classes $\mathcal{C}$ and $\mathcal{C}^*$ are closed under rational equivalence.

LEMMA 2.9. *Let systems $S = 1$ and $U = 1$ be rationally equivalent. Then:*

(1) *If $U = 1$ is in $\mathcal{C}$ then $S = 1$ is $\mathcal{C}$;*
(2) *If $U = 1$ is in $\mathcal{C}^*$ then $S = 1$ is $\mathcal{C}^*$.*

PROOF. We prove (2), a similar argument proves (1). Suppose that a system $S(X) = 1$ is rationally equivalent to a system $U(Z) = 1$ from $\mathcal{C}^*$. Then (see [**2**]) their coordinate groups $F_{R(S)}$ and $F_{R(U)}$ are F-isomorphic. Let $\phi : F_{R(S)} \to F_{R(U)}$ be an F-isomorphism. Then $X^\phi = P(Z)$ for some word mapping P. Suppose now that a formula

$$T(X, Y) = 1 \wedge W(X, Y) \neq 1$$

is compatible with $S(X) = 1$ over F. One needs to show that this formula admits an S-lift. Notice that

$$T(P(Z), Y) = 1 \wedge W(P(Z), Y) \neq 1$$

is compatible with $U(Z) = 1$, hence it admits a U-lift. So there exists an element, say $D(Z) \in F_{R(U)}$, such that in $F_{R(U)}$ the following holds

$$T(P(Z), D(Z)) = 1 \wedge W(P(Z), D(Z)) \neq 1.$$

Now

$$1 = T(P(Z), D(Z))^{\phi^{-1}} = T(P(Z)^{\phi^{-1}}, D(Z^{\phi^{-1}})) = T(X, D(Z^{\phi^{-1}}))$$

and

$$1 \neq W(P(Z), D(Z))^{\phi^{-1}} = W(X, D(Z^{\phi^{-1}}))$$

so

$$T(P(Z), Y) = 1 \wedge W(P(Z), Y) \neq 1$$

admits a complete S-lift, as required. $\square$

3. Cut equations

We refer to [**16**] for the notion of a generalized equation. In the proof of the implicit function theorems it will be convenient to use a modification of the notion of a generalized equation. The following definition provides a framework for such a modification.

Definition 3.1. A cut equation $\Pi = (\mathcal{E}, M, X, f_M, f_X)$ consists of a set of intervals $\mathcal{E}$, a set of variables M, a set of parameters X, and two labeling functions

$$f_X : \mathcal{E} \to F[X], \quad f_M : \mathcal{E} \to F[M].$$

For an interval $\sigma \in \mathcal{E}$ the image $f_M(\sigma) = f_M(\sigma)(M)$ is a reduced word in variables $M^{\pm 1}$ and constants from F, we call it a *partition* of $f_X(\sigma)$.

Sometimes we write $\Pi = (\mathcal{E}, f_M, f_X)$ omitting M and X.

Definition 3.2. A solution of a cut equation $\Pi = (\mathcal{E}, f_M, f_X)$ with respect to an F-homomorphism $\beta : F[X] \to F$ is an F-homomorphism $\alpha : F[M] \to F$ such that: 1) for every $\mu \in M$ $\alpha(\mu)$ is a reduced non-empty word; 2) for every reduced word $f_M(\sigma)(M)$ $(\sigma \in \mathcal{E})$ the replacement $m \to \alpha(m)$ $(m \in M)$ results in a word $f_M(\sigma)(\alpha(M))$ which is a reduced word as written and such that $f_M(\sigma)(\alpha(M))$ is graphically equal to the reduced form of $\beta(f_X(\sigma))$; in particular, the following diagram is commutative.

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, "f_X"'] \arrow[dr, "f_M"] & \\ F(X) \arrow[dr, "\beta"'] & & F(M) \arrow[dl, "\alpha"] \\ & F & \end{tikzcd}$$

If $\alpha : F[M] \to F$ is a solution of a cut equation $\Pi = (\mathcal{E}, f_M, f_X)$ with respect to an F-homomorphism $\beta : F[X] \to F$, then we write (Π, β, α) and refer to α as a *solution of* Π *modulo* β. In this event, for a given $\sigma \in \mathcal{E}$ we say that $f_M(\sigma)(\alpha(M))$ is a *partition* of $\beta(f_X(\sigma))$. Sometimes we also consider homomorphisms $\alpha : F[M] \to F$, for which the diagram above is still commutative, but cancellation may occur in the words $f_M(\sigma)(\alpha(M))$. In this event we refer to α as a *group* solution of Π with respect to β.

Lemma 3.3. *For a generalized equation $\Omega(H)$ one can effectively construct a cut equation $\Pi_\Omega = (\mathcal{E}, f_X, f_M)$ such that the following conditions hold:*

(1) *X is a partition of the whole interval $[1, \rho_\Omega]$ into disjoint closed subintervals;*
(2) *M contains the set of variables H;*
(3) *for any solution $U = (u_1, \ldots, u_\rho)$ of Ω the cut equation Π_Ω has a solution α modulo the canonical homomorphism*

$$\beta_U : F(X) \to F$$

($\beta_U(x) = u_i u_{i+1} \cdots u_j$ where i, j are, correspondingly, the left and the right end-points of the interval x);
(4) *for any solution (β, α) of the cut equation Π_Ω the restriction of α on H gives a solution of the generalized equation Ω.*

PROOF. We begin with defining the sets X and M. Recall that a closed interval of Ω is a union of closed sections of Ω. Let X be an arbitrary partition of the whole interval $[1, \rho_\Omega]$ into closed subintervals (i.e., any two intervals in X are disjoint and the union of X is the whole interval $[1, \rho_\Omega]$).

Let B be a set of representatives of dual bases of Ω, i.e., for every base μ of Ω either μ or $\Delta(\mu)$ belongs to B, but not both. Put $M = H \cup B$.

Now let $\sigma \in X$. We denote by B_σ the set of all bases over σ and by H_σ the set of all items in σ. Put $S_\sigma = B_\sigma \cup H_\sigma$. For $e \in S_\sigma$ let $s(e)$ be the interval $[i, j]$, where $i < j$ are the endpoints of e. A sequence $P = (e_1, \ldots, e_k)$ of elements from S_σ is called a *partition* of σ if $s(e_1) \cup \cdots \cup s(e_k) = \sigma$ and $s(e_i) \cap s(e_j) = \emptyset$ for $i \neq j$. Let Part_σ be the set of all partitions of σ. Now put

$$\mathcal{E} = \{P \mid P \in \mathrm{Part}_\sigma, \sigma \in X\}.$$

Then for every $P \in \mathcal{E}$ there exists one and only one $\sigma \in X$ such that $P \in \mathrm{Part}_\sigma$. Denote this σ by $f_X(P)$. The map $f_X : P \to f_X(P)$ is a well-defined function from $\mathcal{E}$ into $F(X)$.

Each partition $P = (e_1, \ldots, e_k) \in \mathrm{Part}_\sigma$ gives rise to a word $w_P(M) = w_1 \ldots w_k$ as follows. If $e_i \in H_\sigma$ then $w_i = e_i$. If $e_i = \mu \in B_\sigma$ then $w_i = \mu^{\varepsilon(\mu)}$. If $e_i = \mu$ and $\Delta(\mu) \in B_\sigma$ then $w_i = \Delta(\mu)^{\varepsilon(\mu)}$. The map $f_M(P) = w_P(M)$ is a well-defined function from $\mathcal{E}$ into $F(M)$.

Now set $\Pi_\Omega = (\mathcal{E}, f_X, f_M)$. It is not hard to see from the construction that the cut equation Π_Ω satisfies all the required properties. Indeed, (1) and (2) follow directly from the construction.

To verify (3), let's consider a solution $U = (u_1, \ldots, u_{\rho_\Omega})$ of Ω. To define corresponding functions β_U and α, observe that the function $s(e)$ (see above) is defined for every $e \in X \cup M$. Now for $\sigma \in X$ put $\beta_U(\sigma) = u_i \ldots u_j$, where $s(\sigma) = [i, j]$, and for $m \in M$ put $\alpha(m) = u_i \ldots u_j$, where $s(m) = [i, j]$. Clearly, α is a solution of Π_Ω modulo β.

To verify (4) observe that if α is a solution of Π_Ω modulo β, then the restriction of α onto the subset $H \subset M$ gives a solution of the generalized equation Ω. This follows from the construction of the words w_p and the fact that the words $w_p(\alpha(M))$ are reduced as written (see definition of a solution of a cut equation). Indeed, if a base μ occurs in a partition $P \in \mathcal{E}$, then there is a partition $P' \in \mathcal{E}$ which is obtained from P by replacing μ by the sequence $h_i \ldots h_j$. Since there is no cancellation in words $w_P(\alpha(M))$ and $w_{P'}(\alpha(M))$, this implies that $\alpha(\mu)^{\varepsilon(\mu)} = \alpha(h_i \ldots h_j)$. This shows that α_H is a solution of Ω. $\square$

THEOREM 3.4. *Let $S(X,Y,A)) = 1$ be a system of equations over $F = F(A)$. Then one can effectively construct a finite set of cut equations*

$$\mathcal{C}E(S) = \{\Pi_i \mid \Pi_i = (\mathcal{E}_i, f_{X_i}, f_{M_i}), i = 1\ldots,k\}$$

and a finite set of tuples of words $\{Q_i(M_i) \mid i = 1,\ldots,k\}$ such that:

(1) *for every equation $\Pi_i = (\mathcal{E}_i, f_{X_i}, f_{M_i}) \in \mathcal{C}E(S)$, one has $X_i = X$ and $f_{X_i}(\mathcal{E}_i) \subset X^{\pm 1}$;*
(2) *for any solution (U,V) of $S(X,Y,A) = 1$ in $F(A)$, there exists a number i and a tuple of words $P_{i,V}$ such that the cut equation $\Pi_i \in \mathcal{C}E(S)$ has a solution $\alpha : M_i \to F$ with respect to the F-homomorphism $\beta_U : F[X] \to F$ which is induced by the map $X \to U$. Moreover, $U = Q_i(\alpha(M_i))$, the word $Q_i(\alpha(M_i))$ is reduced as written, and $V = P_{i,V}(\alpha(M_i))$;*
(3) *for any $\Pi_i \in \mathcal{C}E(S)$ there exists a tuple of words $P_{i,V}$ such that for any solution (group solution) (β,α) of Π_i the pair (U,V), where $U = Q_i(\alpha(M_i))$ and $V = P_{i,V}(\alpha(M_i))$, is a solution of $S(X,Y) = 1$ in F.*

PROOF. Let $S(X,Y) = 1$ be a system of equations over a free group F. In [**16**, Section 4.4] we have constructed a set of initial parameterized generalized equations $\mathcal{G}E_{\rm par}(S) = \{\Omega_1,\ldots,\Omega_r\}$ for $S(X,Y) = 1$ with respect to the set of parameters X. For each $\Omega \in \mathcal{G}E_{\rm par}(S)$ in [**16**, Section 8] we constructed the finite tree $T_{\rm sol}(\Omega)$ with respect to parameters X. Observe that parametric part $[j_{v_0}, \rho_{v_0}]$ in the root equation $\Omega = \Omega_{v_0}$ of the tree $T_{\rm sol}(\Omega)$ is partitioned into a disjoint union of closed sections corresponding to X-bases and constant bases (this follows from the construction of the initial equations in the set $\mathcal{G}E_{\rm par}(S)$). We label every closed section σ corresponding to a variable $x \in X^{\pm 1}$ by x, and every constant section corresponding to a constant a by a. Due to our construction of the tree $T_{\rm sol}(\Omega)$ moving along a branch B from the initial vertex v_0 to a terminal vertex v, we transfer all the bases from the active and non-active parts into parametric parts until, eventually, in Ω_v the whole interval consists of the parametric part. Observe also that, moving along B in the parametric part, we neither introduce new closed sections nor delete any. All we do is we split (sometimes) an item in a closed parametric section into two new ones. In any event we keep the same label of the section.

Now for a terminal vertex v in $T_{\rm sol}(\Omega)$ we construct a cut equation $\Pi'_v = (\mathcal{E}_v, f_{X_v}, f_{M_v})$ as in Lemma 3.3 taking the set of all closed sections of Ω_v as the partition X_v. The set of cut equations

$$\mathcal{C}E'(S) = \{\Pi'_v \mid \Omega \in \mathcal{G}E_{\rm par}(S), v \in VTerm(T_{\rm sol}(\Omega))\}$$

satisfies all the requirements of the theorem except X_v might not be equal to X. To satisfy this condition we adjust slightly the equations Π'_v.

To do this, we denote by $l : X_v \to X^{\pm 1} \cup A^{\pm 1}$ the labelling function on the set of closed sections of Ω_v. Put $\Pi_v = (\mathcal{E}_v, f_X, f_{M_v})$ where f_X is the composition of f_{X_v} and l. The set of cut equations

$$\mathcal{C}E(S) = \{\Pi_v \mid \Omega \in \mathcal{G}E_{\rm par}(S), v \in VTerm(T_{\rm sol}(\Omega))\}$$

satisfies all the conditions of the theorem. This follows from [**16**, Theorem 8.1], and from Lemma 3.3. Indeed, to satisfy 3) one can take the words $P_{i,V}$ that correspond to a minimal solution of Π_i, i.e., the words $P_{i,V}$ can be obtained from a given particular way to transfer all bases from Y-part onto X-part.

$\square$

The next result shows that for every cut equation Π one can effectively and canonically associate a generalized equation Ω_Π.

For every cut equation $\Pi = (\mathcal{E}, X, M, f_X, f_M)$ one can canonically associate a generalized equation $\Omega_\Pi(M, X)$ as follows. Consider the following word

$$V = f_X(\sigma_1) f_M(\sigma_1) \cdots f_X(\sigma_k) f_M(\sigma_k).$$

Now we are going to mimic the construction of the generalized equation in [**16**, Lemma 4.6]. The set of boundaries BD of Ω_Π consists of positive integers $1, \ldots, |V|+1$. The set of bases BS is union of the following sets:

a) every letter μ in the word V. Letters $X^{\pm 1} \cup M^{\pm 1}$ are variable bases, for every two different occurrences $\mu^{\varepsilon_1}, \mu^{\varepsilon_2}$ of a letter $\mu \in X^{\pm 1} \cup M^{\pm 1}$ in V we say that these bases are dual and they have the same orientation if $\varepsilon_1 \varepsilon_2 = 1$, and different orientation otherwise. Each occurrence of a letter $a \in A^{\pm 1}$ provides a constant base with the label a. Endpoints of these bases correspond to their positions in the word V [**16**, Lemma 4.6].

b) every pair of subwords $f_X(\sigma_i), f_M(\sigma_i)$ provides a pair of dual bases $\lambda_i, \Delta(\lambda_i)$, the base λ_i is located above the subword $f_X(\sigma_i)$, and $\Delta(\lambda_i)$ is located above $f_M(\sigma_i)$ (this defines the endpoints of the bases).

Informally, one can visualize the generalized equation Ω_Π as follows. Let $\mathcal{E} = \{\sigma_1, \ldots, \sigma_k\}$ and let $\mathcal{E}' = \{\sigma' \mid \sigma \in \mathcal{E}\}$ be another disjoint copy of the set $\mathcal{E}$. Locate intervals from $\mathcal{E} \cup \mathcal{E}'$ on a segment I of a straight line from left to the right in the following order $\sigma_1, \sigma_1', \ldots, \sigma_k, \sigma_k'$; then put bases over I according to the word V. The next result summarizes the discussion above.

LEMMA 3.5. *For every cut equation $\Pi = (\mathcal{E}, X, M, f_x, f_M)$, one can canonically associate a generalized equation $\Omega_\Pi(M, X)$ such that if $\alpha_\beta : F[M] \to F$ is a solution of the cut equation Π, then the maps $\alpha : F[M] \to F$ and $\beta : F[X] \to F$ give rise to a solution of the group equation (not generalized!) $\Omega_\Pi^* = 1$ in such a way that for every $\sigma \in \mathcal{E}$ $f_M(\sigma)(\alpha(M))$ is a reduced word which is graphically equal to $\beta(f_X(\sigma)(X))$, and vice versa.*

4. Basic automorphisms of orientable quadratic equations

In this section, for a finitely generated fully residually free group G we introduce some particular G-automorphisms of a free G-group $G[X]$ which fix a given standard orientable quadratic word with coefficients in G. Then we describe some cancellation properties of these automorphisms.

Let G be a group and let $S(X) = 1$ be a regular standard orientable quadratic equation over G :

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] d^{-1} = 1, \tag{7}$$

where c_i, d are non-trivial constants from G, and

$$X = \{x_i, y_i, z_j \mid i = 1, \ldots, n, j = 1, \ldots, m\}$$

is the set of variables. Sometimes we omit X and write simply $S = 1$. Denote by

$$C_S = \{c_1, \ldots, c_m, d\}$$

the set of constants which occur in the equation $S = 1$.

Below we define a *basic sequence*

$$\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{K(m,n)})$$

of G-automorphisms of the free G-group $G[X]$, each of which fixes the element

$$S_0 = \prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] \in G[X].$$

We assume that each $\gamma \in \Gamma$ acts identically on all the generators from X that are not mentioned in the description of γ.

Let $m \geqslant 1, n = 0$. In this case $K(m,0) = m-1$. Put

$$\gamma_i \quad : \quad z_i \to z_i(c_i^{z_i} c_{i+1}^{z_{i+1}}), \quad z_{i+1} \to z_{i+1}(c_i^{z_i} c_{i+1}^{z_{i+1}}), \quad \text{for } i = 1, \ldots, m-1.$$

Let $m = 0$, $n \geqslant 1$. In this case $K(0,n) = 4n-1$. Put

$$\begin{aligned}
\gamma_{4i-3} \quad &: \quad y_i \to x_i y_i, \quad \text{for } i = 1, \ldots, n;\\
\gamma_{4i-2} \quad &: \quad x_i \to y_i x_i, \quad \text{for } i = 1, \ldots, n;\\
\gamma_{4i-1} \quad &: \quad y_i \to x_i y_i, \quad \text{for } i = 1, \ldots, n;\\
\gamma_{4i} \quad &: \quad x_i \to (y_i x_{i+1}^{-1})^{-1} x_i, \quad y_i \to y_i^{y_i x_{i+1}^{-1}}, \quad x_{i+1} \to x_{i+1}^{y_i x_{i+1}^{-1}},\\
&\quad\; y_{i+1} \to (y_i x_{i+1}^{-1})^{-1} y_{i+1}\ , \text{ for } i = 1, \ldots, n-1.
\end{aligned}$$

Let $m \geqslant 1$, $n \geqslant 1$. In this case $K(m,n) = m + 4n - 1$. Put

$$\begin{aligned}
\gamma_i \quad &: \quad z_i \to z_i(c_i^{z_i} c_{i+1}^{z_{i+1}}), \quad z_{i+1} \to z_{i+1}(c_i^{z_i} c_{i+1}^{z_{i+1}}), \quad \text{for } i = 1, \ldots, m-1;\\
\gamma_m \quad &: \quad z_m \to z_m(c_m^{z_m} x_1^{-1}), \quad x_1 \to x_1^{c_m^{z_m} x_1^{-1}}, \quad y_1 \to (c_m^{z_m} x_1^{-1})^{-1} y_1;\\
\gamma_{m+4i-3} \quad &: \quad y_i \to x_i y_i, \quad \text{for } i = 1, \ldots, n;\\
\gamma_{m+4i-2} \quad &: \quad x_i \to y_i x_i, \quad \text{for } i = 1, \ldots, n;\\
\gamma_{m+4i-1} \quad &: \quad y_i \to x_i y_i, \quad \text{for } i = 1, \ldots, n;\\
\gamma_{m+4i} \quad &: \quad x_i \to (y_i x_{i+1}^{-1})^{-1} x_i, \quad y_i \to y_i^{y_i x_{i+1}^{-1}}, \quad x_{i+1} \to x_{i+1}^{y_i x_{i+1}^{-1}},\\
&\quad\; y_{i+1} \to (y_i x_{i+1}^{-1})^{-1} y_{i+1}, \text{ for } i = 1, \ldots, n-1.
\end{aligned}$$

It is easy to check that each $\gamma \in \Gamma$ fixes the word S_0 as well as the word S. This shows that γ induces a G-automorphism on the group $G_S = G[X]/\mathrm{ncl}(S)$. We denote the induced automorphism again by γ, so $\Gamma \subset Aut_G(G_S)$. Since $S = 1$ is regular, $G_S = G_{R(S)}$. It follows that composition of any product of automorphisms from Γ and a particular solution β of $S = 1$ is again a solution of $S = 1$.

Observe, that in the case $m \neq 0, n \neq 0$ the basic sequence of automorphisms Γ contains the basic automorphisms from the other two cases. This allows us, without loss of generality, to formulate some of the results below only for the case $K(m,n) = m + 4n - 1$. Obvious adjustments provide the proper argument in the other cases. From now on we order elements of the set X in the following way

$$z_1 < \ldots < z_m < x_1 < y_1 < \ldots < x_n < y_n.$$

For a word $w \in F(X)$ we denote by $v(w)$ the *leading* variable (the highest variable with respect to the order introduced above) that occurs in w. For $v = v(w)$ denote by $j(v)$ the following number

$$j(v)=\begin{cases} m+4i, & \text{if } v=x_i \text{ or } v=y_i \text{ and } i<n,\\ m+4i-1, & \text{if } v=x_i \text{ or } v=y_i \text{ and } i=n,\\ i, & \text{if } v=z_i \text{ and } n\neq 0,\\ m-1, & \text{if } v=z_m, \text{ n= } 0.\end{cases}$$

The following lemma describes the action of powers of basic automorphisms from Γ on X. The proof is obvious, and we omit it.

LEMMA 4.1. *Let $\Gamma=(\gamma_1,\ldots,\gamma_{m+4n-1})$ be the basic sequence of automorphisms and p be a positive integer. Then the following hold:*

$$\begin{array}{lcl}
\gamma_i^p & : & z_i\to z_i(c_i^{z_i}c_{i+1}^{z_{i+1}})^p, \quad z_{i+1}\to z_{i+1}(c_i^{z_i}c_{i+1}^{z_{i+1}})^p,\\
& & \qquad\qquad \textit{for } \ i=1,\ldots,m-1;\\
\gamma_m^p & : & z_m\to z_m(c_m^{z_m}x_1^{-1})^p, \quad x_1\to x_1^{(c_m^{z_m}x_1^{-1})^p}, \quad y_1\to (c_m^{z_m}x_1^{-1})^{-p}y_1;\\
\gamma_{m+4i-3}^p & : & y_i\to x_i^p y_i, \quad \textit{for } \ i=1,\ldots,n;\\
\gamma_{m+4i-2}^p & : & x_i\to y_i^p x_i, \quad \textit{for } \ i=1,\ldots,n;\\
\gamma_{m+4i-1}^p & : & y_i\to x_i^p y_i, \quad \textit{for } \ i=1,\ldots,n;\\
\gamma_{m+4i}^p & : & x_i\to (y_ix_{i+1}^{-1})^{-p}x_i, \ y_i\to y_i^{(y_ix_{i+1}^{-1})^p},\\
& & x_{i+1}\to x_{i+1}^{(y_ix_{i+1}^{-1})^p}, \quad y_{i+1}\to (y_ix_{i+1}^{-1})^{-p}y_{i+1},\\
& & \qquad\qquad \textit{for } \ i=1,\ldots,n-1.
\end{array}$$

The p-powers of elements that occur in Lemma 4.1 play an important part in what follows, so we describe them in a separate definition.

DEFINITION 4.2. Let $\Gamma=(\gamma_1,\ldots,\gamma_{m+4n-1})$ be the basic sequence of automorphism for $S=1$. For every $\gamma\in\Gamma$ we define the leading term $A(\gamma)$ as follows:

$A(\gamma_i)=c_i^{z_i}c_{i+1}^{z_{i+1}}$, for $i=1,\ldots,m-1$;

$A(\gamma_m)=c_m^{z_m}x_1^{-1}$;

$A(\gamma_{m+4i-3})=x_i$, for $i=1,\ldots,n$;

$A(\gamma_{m+4i-2})=y_i$, for $i=1,\ldots,n$;

$A(\gamma_{m+4i-1})=x_i$, for $i=1,\ldots,n$;

$A(\gamma_{m+4i})=y_ix_{i+1}^{-1}$, for $i=1,\ldots,n-1$.

Now we introduce vector notations for automorphisms of particular type.

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}^k$ the set of all k-tuples of elements from $\mathbb{N}$. For $s\in\mathbb{N}$ and $p\in\mathbb{N}^k$ we say that the tuple p is *s-large* if every coordinate of p is greater then s. Similarly, a subset $P\subset\mathbb{N}^k$ is *s-large* if every tuple in P is s-large. We say that the set P is *unbounded* if for any $s\in\mathbb{N}$ there exists an s-large tuple in P.

Let $\delta=(\delta_1,\ldots,\delta_k)$ be a sequence of G-automorphisms of the group $G[X]$, and $p=(p_1,\ldots,p_k)\in\mathbb{N}^k$. Then by δ^p we denote the following automorphism of $G[X]$:

$$\delta^p=\delta_1^{p_1}\cdots\delta_k^{p_k}.$$

NOTATION 4.3. Let $\Gamma=(\gamma_1,\ldots,\gamma_K)$ be the basic sequence of automorphisms for $S=1$. Denote by Γ_∞ the infinite periodic sequence with period Γ, i.e., $\Gamma_\infty=$

$\{\gamma_i\}_{i\geqslant 1}$ with $\gamma_{i+K} = \gamma_i$. For $j \in \mathbb{N}$ denote by Γ_j the initial segment of Γ_∞ of length j. Then for a given j and $p \in \mathbb{N}^j$ put

$$\phi_{j,p} = \overleftarrow{\Gamma}_j^{\overleftarrow{p}} = \gamma_j^{p_j}\gamma_{j-1}^{p_{j-1}}\cdots\gamma_1^{p_1}.$$

Sometimes we omit p from $\phi_{j,p}$ and write simply ϕ_j.

Agreement. *From now on we fix an arbitrary positive multiple L of the number $K = K(m,n)$, a 2-large tuple $p \in \mathbb{N}^L$, and the automorphism $\phi = \phi_{L,p}$ (as well as all the automorphism ϕ_j, $j \leqslant L$).*

Definition 4.4. The leading term $A_j = A(\phi_j)$ of the automorphism ϕ_j is defined to be the cyclically reduced form of the word

$$\begin{cases} A(\gamma_j)^{\phi_{j-1}}, & \text{if } j \neq m+4i-1+sK \text{ for any } i = 1,\dots,n, s\in\mathbb{N};, \\ y_i^{-\phi_{j-2}}A(\gamma_j)^{\phi_{j-1}}y_i^{\phi_{j-2}}, & \text{if } j = m+4i-1+sK \text{ for some } i = 1,\dots,n, s\in\mathbb{N}. \end{cases}$$

Lemma 4.5. *For every $j \leqslant L$ the element A_j is not a proper power in $G[X]$.*

Proof. It is easy to check that $A(\gamma_s)$ from Definition 4.2 is not a proper power for $s = 1,\dots,K$. Since $A_j)$ is the image of some $A(\gamma_s)$ under an automorphism of $G[X]$ it is not a proper power in $G[X]$. $\square$

For words $w, u, v \in G[X]$, the notation

$$\left|\begin{array}{c} w \\ \hline u \quad v \end{array}\right|$$

means that $w = u \circ w' \circ v$ for some $w' \in G[X]$, where the length of elements and reduced form defined as in the free product $G * \langle X\rangle$. Similarly, notations $\left|\begin{array}{l} w \\ \hline u \end{array}\right.$ and $\left.\begin{array}{r} w \\ \hline v \end{array}\right|$ mean that $w = u\circ w'$ and $w = w'\circ v$. Sometimes we write $\left|\begin{array}{c} w \\ \hline u \quad * \end{array}\right|$ or $\left|\begin{array}{c} w \\ \hline * \quad v \end{array}\right|$ when the corresponding words are irrelevant.

If n is a positive integer and $w \in G[X]$, then by $Sub_n(w)$ we denote the set of all n-subwords of w, i.e.,

$$Sub_n(w) = \{u \mid |u| = n \text{ and } w = w_1\circ u\circ w_2 \text{ for some } w_1, w_2 \in G[X]\}.$$

Similarly, by $SubC_n(w)$ we denote all n-subwords of the *cyclic* word w. More generally, if $W \subseteq G[X]$, then

$$Sub_n(W) = \bigcup_{w\in W} Sub_n(w), \quad SubC_n(W) = \bigcup_{w\in W} SubC_n(w).$$

Obviously, the set $Sub_i(w)$ ($SubC_i(w)$) can be effectively reconstructed from $Sub_n(w)$ ($SubC_n(w)$) for $i \leqslant n$.

In the following series of lemmas we write down explicit expressions for images of elements of X under the automorphism

$$\phi_K = \gamma_K^{p_K}\cdots\gamma_1^{p_1}, \quad K = K(m,n).$$

These lemmas are very easy and straightforward, though tiresome in terms of notations. They provide basic data needed to prove the implicit function theorem. All elements that occur in the lemmas below can be viewed as elements (words) from the free group $F(X\cup C_S)$. In particular, the notations $\circ$, $\left|\begin{array}{c} w \\ \hline u \quad v \end{array}\right|$, and $Sub_n(W)$

correspond to the standard length function on $F(X \cup C_S)$. Furthermore, until the end of this section we assume that the elements $c_1, \ldots, c_m$ are *pairwise different*.

LEMMA 4.6. *Let $m \neq 0$, $K = K(m,n)$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, and*

$$\phi_K = \gamma_K^{p_K} \cdots \gamma_1^{p_1}.$$

The following statements hold.

(1) *All automorphisms from Γ, except for γ_{i-1}, γ_i (if defined), fix z_i, $i = 1, \ldots, m$. It follows that*

$$z_i^{\phi_K} = \ldots = z_i^{\phi_i}$$

for $i = 1, \ldots, m-1$.

(2) *Let $\tilde{z}_i = z_i^{\phi_{i-1}}$ $(i = 2, \ldots, m)$, $\tilde{z}_1 = z_1$. Then*

$$\tilde{z}_i = \left|\begin{array}{c} z_i \circ (c_{i-1}^{\tilde{z}_{i-1}} \circ c_i^{z_i})^{p_{i-1}} \\ \hline {\scriptstyle z_i z_{i-1}^{-1} \qquad\qquad c_i z_i} \end{array}\right|$$

for $i = 2, \ldots, m$.

(3) *The reduced forms of the leading terms of the corresponding automorphisms are listed below:*

$$\begin{array}{rcl}
A_1 & = & \left|\begin{array}{c} c_1^{z_1} \circ c_2^{z_2} \\ \hline {\scriptstyle z_1^{-1}c_1 \quad c_2 z_2} \end{array}\right|, \\
 & & A_2 = A_1^{-p_1} c_2^{z_2} A_1^{p_1} c_3^{z_3}, (m \geqslant 2) \\
SubC_3(A_1) & = & \{z_1^{-1}c_1z_1,\ c_1z_1z_2^{-1},\ z_1z_2^{-1}c_2,\ z_2^{-1}c_2z_2,\ c_2z_2z_1^{-1},\ z_2z_1^{-1}c_1\}; \\
A_i & = & \left|\begin{array}{c} A_{i-1}^{-p_{i-1}} \\ \hline {\scriptstyle z_i^{-1}c_i^{-1} \quad c_{i-1}z_{i-1}} \end{array}\right| c_i^{z_i} \left|\begin{array}{c|c} A_{i-1}^{p_{i-1}} & c_{i+1}^{z_{i+1}} \\ \hline {\scriptstyle z_{i-1}^{-1}c_{i-1}^{-1} \quad c_i z_i} & {\scriptstyle z_{i+1}^{-1} \quad c_{i+1}z_{i+1}} \end{array}\right|, \\
 & & i = 3, \ldots, m-1, \\
SubC_3(A_i) & = & SubC_3(A_{i-1})^{\pm 1} \\
 & & \cup\{c_{i-1}z_{i-1}z_i^{-1},\ z_{i-1}z_i^{-1}c_i,\ z_i^{-1}c_iz_i,\ c_iz_iz_{i-1}^{-1}, z_iz_{i-1}^{-1}c_{i-1}^{-1}, \\
 & & c_iz_iz_{i+1}^{-1},\ z_iz_{i+1}^{-1}c_{i+1},\ z_{i+1}^{-1}c_{i+1}z_{i+1},\ c_{i+1}z_{i+1}z_i^{-1},\ z_{i+1}z_i^{-1}c_i^{-1}\}; \\
A_2 & = & A_1^{-p_1} c_2^{z_2} A_1^{p_1} x_1^{-1} (m = 2); \\
A_m & = & \left|\begin{array}{c} A_{m-1}^{-p_{m-1}} \\ \hline {\scriptstyle z_m^{-1}c_m^{-1} \quad c_{m-1}z_{m-1}} \end{array}\right| c_m^{z_m} \left|\begin{array}{c} A_{m-1}^{p_{m-1}} \\ \hline {\scriptstyle z_{m-1}^{-1}c_{m-1}^{-1} \quad c_m z_m} \end{array}\right| x_1^{-1} \\
 & & (n \neq 0, m > 2), \\
SubC_3(A_m) & = & SubC_3(A_{m-1})^{\pm 1} \\
 & & \cup\{c_{m-1}z_{m-1}z_m^{-1},\ z_{m-1}z_m^{-1}c_m,\ z_{m-1}^{-1}c_mz_m, \\
 & & c_mz_mz_{m-1}^{-1},\ c_mz_mx_1^{-1}, z_mx_1^{-1}z_m^{-1},\ x_1^{-1}z_m^{-1}c_m^{-1}\}.
\end{array}$$

(4) *The reduced forms of $z_i^{\phi_{i-1}}, z_i^{\phi_i}$ are listed below:*

$$
\begin{aligned}
z_1^{\phi_K} &= z_1^{\phi_1} = c_1 \underset{z_1z_2^{-1}\qquad c_2z_2}{\left| z_1c_2^{z_2} \right|} \underset{z_1^{-1}c_1\qquad c_2z_2}{\left| A_1^{p_1-1} \right|} \quad (m \geq 2)\ ,\\
SubC_3(z_1^{\phi_K}) &= \{\ c_1z_1z_2^{-1},\ z_1z_2^{-1}c_2,\ z_2^{-1}c_2z_2,\ c_2z_2z_1^{-1},\ z_2z_1^{-1}c_1,\ z_1^{-1}c_1z_1\};\\
z_i^{\phi_{i-1}} &= z_i \underset{z_{i-1}^{-1}c_{i-1}^{-1}\qquad c_iz_i}{\left| A_{i-1}^{p_{i-1}} \right|}\ ,\\
z_i^{\phi_K} &= z_i^{\phi_i} = c_iz_i \underset{z_{i-1}^{-1}c_{i-1}^{-1}\qquad c_iz_i}{\left| A_{i-1}^{p_{i-1}} \right|} c_{i+1}^{z_{i+1}} \underset{z_i^{-1}c_i^{-1}\qquad c_{i+1}z_{i+1}}{\left| A_i^{p_i-1} \right|}\\
&\qquad (i = 3, \dots, m-1),\\
Sub_3(z_i^{\phi_K}) &= SubC_3(A_{i-1}) \cup SubC_3(A_i) \cup \{c_iz_iz_{i-1}^{-1},\ z_iz_{i-1}^{-1}c_{i-1}^{-1},\\
&\quad c_iz_iz_{i+1}^{-1},\ z_iz_{i+1}^{-1}c_{i+1},\ z_{i+1}^{-1}c_{i+1}z_{i+1},\ c_{i+1}z_{i+1}z_i^{-1},\ z_{i+1}z_i^{-1}c_i^{-1}\}\ ;\\
z_m^{\phi_K} &= z_m^{\phi_{m-1}} = z_m \underset{z_{m-1}^{-1}c_{m-1}^{-1}\qquad c_mz_m}{\left| A_{m-1}^{p_{m-1}} \right|}\ , \quad (n = 0),\\
Sub_3(z_m^{\phi_K}) &= SubC_3(A_{m-1}) \cup \{c_mz_mz_{m-1}^{-1},\ z_mz_{m-1}^{-1}c_{m-1}^{-1}\}(\textit{when } n = 0)\ ;\\
z_m^{\phi_K} &= c_mz_m^{\phi_m} = c_mz_m \underset{z_{m-1}^{-1}c_{m-1}^{-1}\qquad c_mz_m}{\left| A_{m-1}^{p_{m-1}} \right|} x_1^{-1} \underset{z_m^{-1}c_m^{-1}\qquad z_mx_1^{-1}}{\left| A_m^{p_m-1} \right|} \quad (n \neq 0),\\
Sub_3(z_m^{\phi_K}) &= Sub_3(z_m^{\phi_K})_{(when\ n=0)} \cup \{c_mz_mx_1^{-1},\ z_mx_1^{-1}z_m^{-1},\ x_1^{-1}z_m^{-1}c_m^{-1}\}.
\end{aligned}
$$

(5) *The elements* $z_i^{\phi_K}$ *have the following properties:*

$$z_i^{\phi_K} = c_iz_i\hat{z}_i \quad (i = 1, \dots, m-1),$$

where $\hat{z}_i$ *is a word in the alphabet* $\{c_1^{z_1}, \dots, c_{i+1}^{z_{i+1}},\}$ *which begins with* $c_{i-1}^{-z_{i-1}}$, *if* $i \neq 1$, *and with* $c_2^{z_2}$, *if* $i = 1$;

$z_m^{\phi_K} = z_m\hat{z}_m \quad (n = 0)$, *where* $\hat{z}_m$ *is a word in the alphabet* $\{c_1^{z_1}, \dots, c_m^{z_m}\}$;

$z_m^{\phi_K} = c_mz_m\hat{z}_m \quad (n \neq 0)$, *where* $\hat{z}_m$ *is a word in the alphabet*

$$\{c_1^{z_1}, \dots, c_m^{z_m}, x_1\};$$

Moreover, if $m \geq 3$, *the word* $(c_m^{z_m})^{\pm 1}$ *occurs in* $z_i^{\phi_K} \quad (i = m-1, m)$ *only as a part of the subword* $(\prod_{i=1}^m c_i^{z_i})^{\pm 1}$.

Proof. (1) is obvious. We prove (2) by induction. For $i \geqslant 2$,

$$\tilde{z}_i = z_i^{\phi_{i-1}} = z_i^{\gamma_{i-1}^{p_{i-1}}\phi_{i-2}}.$$

Therefore

$$\tilde{z}_i = z_i(c_{i-1}^{\tilde{z}_{i-1}}c_i^{z_i})^{p_{i-1}} = z_i \circ (c_{i-1}^{\tilde{z}_{i-1}} \circ c_i^{z_i})^{p_{i-1}},$$

and the claim follows by induction.

Now we prove (3) and (4) simultaneously. By the straightforward verification one has: $A_1 = \underset{z_1^{-1}\qquad z_2}{\left| c_1^{z_1} \circ c_2^{z_2} \right|}$;

$$z_1^{\phi_1} = z_1^{\gamma_1^{p_1}} = z_1(c_1^{z_1}c_2^{z_2})^{p_1} = \underset{c_1\qquad\qquad z_2}{\left| c_1 \circ z_1 \circ c_2^{z_2} \circ A_1^{p_1-1} \right|}.$$

Denote by cycred (w) the cyclically reduced form of w.

$$A_i = \text{cycred}\ (\left(c_i^{z_i}c_{i+1}^{z_{i+1}}\right)^{\phi_{i-1}}) = \underset{z_i^{-1}\qquad z_{i+1}}{\left| c_i^{\tilde{z}_i} \circ c_{i+1}^{z_{i+1}} \right|} \ (i \leq m-1).$$

Observe that in the notation above

$\tilde{z}_i = z_i A_{i-1}^{p_{i-1}}$ $(i \geq 2)$.

This shows that we can rewrite $A(\phi_i)$ as follows:

$A_i = A_{i-1}^{-p_{i-1}} \circ c_i^{z_i} \circ A_{i-1}^{p_{i-1}} \circ c_{i+1}^{z_{i+1}}$,

beginning with z_i^{-1} and ending with z_{i+1} $\quad (i = 2, \ldots, m-1)$;

$A_m = \text{cycred}\,(c_m^{\tilde{z}_m} x_1^{-1}) = c_m^{\tilde{z}_m} x_1^{-1} = A_{m-1}^{-p_{m-1}} \circ c_m^{z_m} \circ A_{m-1}^{p_{m-1}} \circ x_1^{-1}$ $(m \geq 2)$.

beginning with z_m^{-1} and ending with x_1^{-1} $\quad (n \neq 0)$.

$$z_i^{\phi_{i-1}} = \big(z_i (c_{i-1}^{z_{i-1}} c_i^{z_i})^{p_{i-1}}\big)^{\phi_{i-2}} = z_i (c_{i-1}^{\tilde{z}_{i-1}} c_i^{z_i})^{p_{i-1}} = z_i \circ A_{i-1}^{p_{i-1}},$$

beginning with z_i and ending with z_i;

$$\begin{aligned} z_i^{\phi_i} &= \big(z_i (c_i^{z_i} c_{i+1}^{z_{i+1}})^{p_i}\big)^{\phi_{i-1}} = \tilde{z}_i (c_i^{\tilde{z}_i} c_{i+1}^{z_{i+1}})^{p_i} \\ &= c_i \circ \tilde{z}_i \circ c_{i+1}^{z_{i+1}} \circ (c_i^{\tilde{z}_i} c_{i+1}^{z_{i+1}})^{p_i - 1} \\ &= c_i \circ z_i \circ A_{i-1}^{p_{i-1}} \circ c_{i+1}^{z_{i+1}} \circ A_i^{p_i - 1}, \end{aligned}$$

beginning with c_i and ending with z_{i+1} $\quad (i = 2, \ldots, m-1)$;

$$\begin{aligned} z_m^{\phi_m} &= \big(z_m (c_m^{z_m} x_1^{-1})^{p_m}\big)^{\phi_{m-1}} = \tilde{z}_m (c_m^{\tilde{z}_m} x_1^{-1})^{p_m} \\ &= c_m \tilde{z}_m x_1^{-1} (c_m^{\tilde{z}_m} x_1^{-1})^{p_m - 1} \\ &= c_m \circ z_m \circ A_{m-1}^{p_{m-1}} \circ x_1^{-1} \circ A_m^{p_m - 1} \quad (n \neq 0), \end{aligned}$$

beginning with c_m and ending with x_1^{-1}. This proves the lemma. $\square$

In the following two lemmas we describe the reduced expressions of the elements $x_1^{\phi_K}$ and $y_1^{\phi_K}$.

LEMMA 4.7. *Let $m = 0$, $K = 4n - 1$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, and*

$$\phi_K = \gamma_K^{p_K} \cdots \gamma_1^{p_1}.$$

(1) *All automorphisms from Γ, except for γ_2, γ_4, fix x_1, and all automorphisms from Γ, except for $\gamma_1, \gamma_3, \gamma_4$, fix y_1. It follows that*

$x_1^{\phi_K} = x_1^{\phi_4}, \ y_1^{\phi_K} = y_1^{\phi_4} \quad (n \geqslant 2).$

(2) *Below we list the reduced forms of the leading terms of the corresponding automorphisms (the words on the right are reduced as written)*

$A_1 = x_1$;

$A_2 = x_1^{p_1} y_1 = A_1^{p_1} \circ y_1$;

$A_3 = \underbrace{A_2^{p_2-1}}_{x_1^2 \qquad x_1 y_1} x_1^{p_1+1} y_1,$

$SubC_3(A_3) = SubC_3(A_2) = \{x_1^3, \ x_1^2 y_1, \ x_1 y_1 x_1, y_1 x_1^2\}$;

$A_4 = \underbrace{\Big(\underbrace{A_2^{p_2}}_{x_1^2 \qquad x_1 y_1} x_1\Big)^{p_3}}_{x_1^2 \qquad\qquad y_1 x_1} \underbrace{A_2}_{x_1^2 \qquad x_1 y_1} x_2^{-1} \quad (n \geq 2),$

$SubC_3(A_4) = SubC_3(A_2) \cup \{x_1 y_1 x_2^{-1}, \ y_1 x_2^{-1} x_1, \ x_2^{-1} x_1^2\}.$

(3) *Below we list reduced forms of* $x_1^{\phi_j}, y_1^{\phi_j}$ *for* $j = 1, \ldots, 4$*:*

$x_1^{\phi_1} = x_1;$

$y_1^{\phi_1} = x_1^{p_1} y_1;$

$x_1^{\phi_2} = \underset{x_1^2 \quad x_1 y_1}{\left|\underline{\; A_2^{p_2} \;}\right|} x_1;$

$y_1^{\phi_2} = x_1^{p_1} y_1;$

$x_1^{\phi_3} = x_1^{\phi_2} = \underset{x_1^2 \quad x_1 y_1}{\left|\underline{\; A_2^{p_2} \;}\right|} x_1;$

$Sub_3(x_1^{\phi_K})_{(when \ n=1)} = SubC_3(A_2);$

$y_1^{\phi_3} = \underset{x_1^2 \qquad\qquad x_1 y_1}{\left|\underline{\; (\underset{x_1^2 \quad x_1 y_1}{\left|\underline{\; A_2^{p_2} \;}\right|} x_1)^{p_3} \;}\right|} x_1^{p_1} y_1;$

$Sub_3(y_1^{\phi_K})_{(when \ n=1)} = SubC_3(A_2);$

$x_1^{\phi_4} \;=\; x_1^{\phi_K} \;=\; \underset{x_2 y_1^{-1} \quad x_1^{-2}}{\left|\underline{\; A_4^{-(p_4-1)} \;}\right|} x_2 \underset{y_1^{-1} x_1^{-1} \quad x_1^{-2}}{\left|\underline{\; A_2^{-1} \;}\right|} \underset{x_1^{-1} y_1^{-1} x_1^{-1} \qquad\qquad x_1^{-2}}{\left|\underline{\; (x_1^{-1} \underset{y_1^{-1} x_1^{-1} \quad x_1^{-2}}{\left|\underline{\; A_2^{-p_2} \;}\right|})^{p_3-1} \;}\right|}$

$(n \geqslant 2)$,

$Sub_3(x_1^{\phi_K}) = SubC_3(A_4)^{-1} \cup SubC_3(A_2)^{-1} \cup \{x_1^{-2} x_2, \ x_1^{-1} x_2 y_1^{-1},$
$x_2 y_1^{-1} x_1^{-1}, \ x_1^{-3}, \ x_1^{-2} y_1^{-1}, x_1^{-1} y_1^{-1} x_1^{-1}\} \quad (n \geqslant 2);$

$y_1^{\phi_4} = \underset{x_2 y_1^{-1} \quad x_1^{-1}}{\left|\underline{\; A_4^{-(p_4-1)} \;}\right|} x_2 \underset{x_1^2 \quad y_1 x_2^{-1}}{\left|\underline{\; A_4^{p_4} \;}\right|} \quad (n \geqslant 2),$

$Sub_3(y_1^{\phi_K}) = SubC_3(A_4)^{\pm 1} \cup \{x_1^{-2} x_2, \ x_1^{-1} x_2 x_1, \ x_2 x_1^2\} \quad (n \geqslant 2).$

PROOF. (1) follows directly from definitions.
To show (2) observe that

$A_1 = A(\gamma_1) = x_1;$

$x_1^{\phi_1} = x_1;$

$y_1^{\phi_1} = x_1^{p_1} y_1 = A_1^{p_1} \circ y_1.$

Then

$$\begin{aligned} A_2 &= \text{cycred}(A(\gamma_2)^{\phi_1}) = \text{cycred}(y_1^{\phi_1}) = x_1^{p_1} \circ y_1 = A_1^{p_1} \circ y_1; \\ x_1^{\phi_2} &= (x_1^{\gamma_2^{p_2}})^{\gamma_1^{p_1}} = (y_1^{p_2} x_1)^{\gamma_1^{p_1}} = (x_1^{p_1} y_1)^{p_2} x_1 = A_2^{p_2} \circ x_1; \\ y_1^{\phi_2} &= (y_1^{\gamma_2^{p_2}})^{\gamma_1^{p_1}} = y_1^{\gamma_1^{p_1}} = x_1^{p_1} y_1 = A_2. \end{aligned}$$

Now

$$\begin{aligned} A_3 &= \text{cycred}(y_1^{-\phi_1} A(\gamma_3)^{\phi_2} y_1^{\phi_1}) = \text{cycred}((x_1^{p_1} y_1)^{-1} x_1^{\phi_2} (x_1^{p_1} y_1)) \\ &= \text{cycred}((x_1^{p_1} y_1)^{-1} (x_1^{p_1} y_1)^{p_2} x_1 (x_1^{p_1} y_1)) \\ &= (x_1^{p_1} y_1)^{p_2 - 1} x_1^{p_1+1} y_1 = A_2^{p_2-1} \circ A_1^{p_1+1} \circ y_1. \end{aligned}$$

It follows that

$x_1^{\phi_3} = (x_1^{\gamma_3^{p_3}})^{\phi_2} = x_1^{\phi_2};$

$$y_1^{\phi_3} = (y_1^{\gamma_3^{p_3}})^{\phi_2} = (x_1^{p_3} y_1)^{\phi_2} = (x_1^{\phi_2})^{p_3} y_1^{\phi_2} = (A_2^{p_2} \circ x_1)^{p_3} \circ A_2.$$

Hence

$$A_4 = \operatorname{cycred}(A(\gamma_4)^{\phi_3}) = \operatorname{cycred}(y_1^{\phi_3} x_2^{-\phi_3}) = (A_2^{p_2} \circ x_1)^{p_3} \circ A_2 \circ x_2^{-1}.$$

Finally:

$$\begin{aligned}
x_1^{\phi_4} &= (x_1^{\gamma_4^{p_4}})^{\phi_3} = \big((y_1 x_2^{-1})^{-p_4} x_1\big)^{\phi_3} \\
&= \big((y_1 x_2^{-1})^{\phi_3})\big)^{-p_4} x_1^{\phi_3} = A_4^{-p_4} A_2^{p_2} \circ x_1 \\
&= A_4^{-(p_4-1)} \circ x_2 \circ A_2^{-1} \circ (x_1^{-1} \circ A_2^{-p_2})^{p_3-1} \\
y_1^{\phi_4} &= (y_1^{\gamma_4^{p_4}})^{\phi_3} = (y_1^{(y_1 x_2^{-1})^{p_4}})^{\phi_3} \\
&= \big((y_1 x_2^{-1})^{\phi_3}\big)^{-p_4} y_1^{\phi_3} \big((y_1 x_2^{-1})^{\phi_3}\big)^{p_4} \\
&= A_4^{-p_4} y_1^{\phi_3} A_4^{p_4} = A_4^{-(p_4-1)} A_4^{-1} y_1^{\phi_3} A_4^{p_4} \\
&= A_4^{-(p_4-1)} \circ x_2 \circ A_4^{p_4}.
\end{aligned}$$

This proves the lemma. □

Lemma 4.8. *Let $m \neq 0$, $n \neq 0$, $K = m + 4n - 1$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, and*

$$\phi_K = \gamma_K^{p_K} \cdots \gamma_1^{p_1}.$$

(1) *All automorphisms from Γ except for $\gamma_m, \gamma_{m+2}, \gamma_{m+4}$ fix x_1; and all automorphisms from Γ except for $\gamma_m, \gamma_{m+1}, \gamma_{m+3}, \gamma_{m+4}$ fix y_1. It follows that*

$$x_1^{\phi_K} = x_1^{\phi_{m+4}}, \; y_1^{\phi_K} = y_1^{\phi_{m+4}} \quad (n \geqslant 2).$$

(2) *Below we list the reduced forms of the leading terms of the corresponding automorphisms (the words on the right are reduced as written)*

$$\begin{aligned}
A_{m+1} &= x_1, \\
A_{m+2} &= y_1^{\phi_{m+1}} \\
&= \left|\begin{array}{c} A_m^{-p_m} \\ \hline x_1 z_m^{-1} \quad c_m z_m \end{array}\right| x_1^{p_{m+1}} y_1, \\
SubC_3(A_{m+2}) &= SubC_3(A_m)^{-1} \\
&\qquad \cup \{c_m z_m x_1,\ z_m x_1^2,\ x_1^3,\ x_1^2 y_1,\ x_1 y_1 x_1,\ y_1 x_1 z_m^{-1}\}; \\
A_{m+3} &= \left|\begin{array}{c} A_{m+2}^{p_{m+2}-1} \\ \hline x_1 z_m^{-1} \quad x_1 y_1 \end{array}\right|\begin{array}{c} A_m^{-p_m} \\ \hline x_1 z_m^{-1} \quad c_m z_m \end{array}\right| x_1^{p_{m+1}+1} y_1, \\
SubC_3(A_{m+3}) &= SubC_3(A_{m+2}); \\
A_{m+4} &= \left|\begin{array}{c} A_m^{-p_m} \\ \hline x_1 z_m^{-1} \quad c_m z_m \end{array}\right| \\
&\qquad \circ \left(x_1^{p_{m+1}} y_1 \left|\begin{array}{c} A_{m+2}^{p_{m+2}-1} \\ \hline x_1 z_m^{-1} \quad x_1 y_1 \end{array}\right|\begin{array}{c} A_m^{-p_m} \\ \hline x_1 z_m^{-1} \quad c_m z_m \end{array}\right| x_1 \right)^{p_{m+3}} x_1^{p_{m+1}} y_1 x_2^{-1} \\
&\qquad\quad (n \geqslant 2), \\
SubC_3(A_{m+4}) &= SubC_3(A_{m+2}) \cup \{x_1 y_1 x_2^{-1},\ y_1 x_2^{-1} x_1,\ x_2^{-1} x_1 z_m^{-1}\} \\
&\qquad\quad (n \geqslant 2).
\end{aligned}$$

(3) *Below we list reduced forms of $x_1^{\phi_j}, y_1^{\phi_j}$ for $j = m, \ldots, m+4$ and their expressions via the leading terms:*

$$x_1^{\phi_m} = A_m^{-p_m} \circ x_1 \circ A_m^{p_m},$$

$$y_1^{\phi_m} = A_m^{-p_m} \circ y_1,\ x_1^{\phi_{m+1}} = x_1^{\phi_m},$$

$$y_1^{\phi_{m+1}} = A_m^{-p_m} \circ x_1^{p_{m+1}} \circ y_1,$$

$$x_1^{\phi_{m+2}} = x_1^{\phi_K}{}_{(when\ \ n=1)} = \left|\frac{A_{m+2}^{p_{m+2}}}{\scriptstyle x_1 z_m^{-1} \qquad x_1 y_1}\right|\frac{A_m^{-p_m}}{\scriptstyle x_1 z_m^{-1} \qquad c_m z_m}\left| x_1 \right|\frac{A_m^{p_m}}{\scriptstyle z_m^{-1} c_m^{-1} \qquad z_m x_1^{-1}}\right|,$$

$$Sub_3(x_1^{\phi_K})_{(when\ \ n=1)} = SubC_3(A_{m+2}) \cup SubC_3(A_m) \cup \{z_m x_1 z_m^{-1},\ x_1 z_m^{-1} c_m^{-1}\};$$

$$y_1^{\phi_{m+2}} = y_1^{\phi_{m+1}},$$

$$x_1^{\phi_{m+3}} = x_1^{\phi_{m+2}},$$

$$y_1^{\phi_{m+3}} = y_1^{\phi_K}{}_{(when\ \ n=1)} = \left|\frac{A_m^{-p_m}}{\scriptstyle x_1 z_m^{-1} \qquad c_m z_m}\right|$$
$$\left(x_1^{p_{m+1}} y_1 \left|\frac{A_{m+2}^{p_{m+2}-1}}{\scriptstyle x_1 z_m^{-1} \qquad x_1 y_1}\right|\frac{A_m)^{-p_m}}{\scriptstyle x_1 z_m^{-1} \qquad c_m z_m}\right| x_1 \right)^{p_{m+3}} x_1^{p_{m+1}} y_1.$$

$$Sub_3(y_1^{\phi_K}) =_{(when\ \ n=1)} SubC_3(A_{m+2});$$

$$x_1^{\phi_{m+4}} = x_1^{\phi_K}{}_{(when\ \ n\geqslant 2)} = \left|\frac{A_{m+4}^{-p_{m+4}+1}}{\scriptstyle x_2 y_1^{-1} \qquad z_m x_1^{-1}}\right| x_2 y_1^{-1} x_1^{-p_{m+1}} \circ$$
$$\left(x_1^{-1} \left|\frac{A_m^{p_m}}{\scriptstyle z_m^{-1} c_m^{-1} \qquad z_m x_1^{-1}}\right|\frac{A_{m+2}^{-p_{m+2}}}{\scriptstyle y_1^{-1} x_1^{-1} \qquad z_m x_1^{-1}}\right| y_1^{-1} x_1^{-p_{m+1}} \right)^{p_{m+3}-1} \left|\frac{A_m^{p_m}}{\scriptstyle z_m^{-1} c_m^{-1} \qquad z_m x_1^{-1}}\right| \quad (n \geqslant 2),$$

$$Sub_3(x_1^{\phi_K}) = SubC_3(A_{m+2})^{-1} \cup \{z_m x_1^{-1} x_2,\ x_1^{-1} x_2 y_1^{-1},\ x_2 y_1^{-1} x_1^{-1}\};$$

$$y_1^{\phi_{m+4}} = y_1^{\phi_K}{}_{(when\ \ n\geqslant 2)} = \left|\frac{A_{m+4}^{-(p_{m+4}-1)}}{\scriptstyle x_2 y_1^{-1} \qquad z_m x_1^{-1}}\right| x_2 \left|\frac{A_{m+4}^{p_{m+4}}}{\scriptstyle x_1 z_m^{-1} \qquad y_1 x_2^{-1}}\right| \quad (n \geqslant 2),$$

$$Sub_3(y_1^{\phi_K}) = SubC_3(A_{m+4})^{\pm 1} \cup \{z_m x_1^{-1} x_2,\ x_1^{-1} x_2 x_1, x_2 x_1 z_m^{-1}\}$$
$(n \geqslant 2)$.

Proof. Statement (1) follows immediately from definitions of automorphisms of Γ.

We prove formulas in the second and third statements simultaneously:

$$x_1^{\phi_m} = \left(x_1^{(c_m^{z_m} x_1^{-1})^{p_m}} \right)^{\phi_{m-1}} = x_1^{A_m^{p_m}} = A_m^{-p_m} \circ x_1 \circ A_m^{p_m},$$

beginning with x_1 and ending with x_1^{-1}.

$$y_1^{\phi_m} = \left((c_m^{z_m} x_1^{-1})^{-p_m} y_1 \right)^{\phi_{m-1}} = A_m^{-p_m} \circ y_1,$$

beginning with x_1 and ending with y_1. Now A_{m+1} is the cyclically reduced form of $A(\gamma_{m+1})^{\phi_m} = x_1^{\phi_m} = A_m^{-p_m} \circ x_1 \circ A_m^{p_m}$.

$A_{m+1} = x_1$.

$$\begin{aligned}
x_1^{\phi_{m+1}} &= x_1^{\phi_m}, \\
y_1^{\phi_{m+1}} &= \left(y_1^{\gamma_{m+1}^{p_{m+1}}}\right)^{\phi_m} = (x_1^{p_{m+1}} y_1)^{\phi_m} \\
&= (x_1^{\phi_m})^{p_{m+1}} y_1^{\phi_m} \\
&= A_m^{-p_m} \circ x_1^{p_{m+1}} \circ y_1,
\end{aligned}$$

beginning with x_1 and ending with y_1, moreover, the element that cancels in reducing
$A_{m+1}^{p_{m+1}} A_m^{-p_m} y_1$ is equal to $A_m^{p_m}$.

$$A_{m+2} = \text{cycred}(A(\gamma_{m+2})^{\phi_{m+1}}) = \text{cycred}(y_1^{\phi_{m+1}}) = A_m^{-p_m} \circ x_1^{p_{m+1}} \circ y_1,$$

beginning with x_1 and ending with y_1.

$$\begin{aligned}
x_1^{\phi_{m+2}} &= \left(x_1^{\gamma_{m+2}^{p_{m+2}}}\right)^{\phi_{m+1}} \\
&= (y_1^{\phi_{m+1}})^{p_{m+2}} x_1^{\phi_{m+1}} \\
&= A_{m+2}^{p_{m+2}} \circ A_m^{-p_m} \circ x_1 \circ A_m^{p_m} \\
&= A_m^{-p_m} \circ \left(x_1^{p_{m+1}} \circ y_1 \circ A_{m+2}^{p_{m+2}-1} \circ A_m)^{-p_m} \circ x_1\right) \circ A_m^{p_m},
\end{aligned}$$

beginning with x_1 and ending with x_1^{-1};

$$\begin{aligned}
y_1^{\phi_{m+2}} &= y_1^{\phi_{m+1}}. \\
A_{m+3} &= y_1^{-\phi_{m+1}} x_1^{\phi_{m+2}} y_1^{\phi_{m+1}} \\
&= A_{m+2}^{p_{m+2}-1} \circ A_m^{-p_m} \circ x_1^{p_{m+1}+1} \circ y_1,
\end{aligned}$$

beginning with x_1 and ending with y_1;

$$\begin{aligned}
x_1^{\phi_{m+3}} &= x_1^{\phi_{m+2}}, \\
y_1^{\phi_{m+3}} &= (x_1^{\phi_{m+2}})^{p_{m+3}} y_1^{\phi_{m+1}} \\
&= A_m^{-p_m} \circ \left(x_1^{p_{m+1}} \circ y_1 \circ A_{m+2}^{p_{m+2}-1} \circ A_m^{-p_m} \circ x_1\right)^{p_{m+3}} \circ x_1^{p_{m+1}} \circ y_1,
\end{aligned}$$

beginning with x_1 and ending with y_1. Finally,

$$A_{m+4} = \text{cycred}(A(\gamma_{m+4})^{\phi_{m+3}}) = \text{cycred}((y_1 x_2^{-1})^{\phi_{m+3}}) = y_1^{\phi_{m+3}} \circ x_2^{-1},$$

beginning with x_1 and ending with x_2^{-1};

$$\begin{aligned}
x_1^{\phi_{m+4}} &= \left((y_1 x_2^{-1})^{-p_{m+4}} x_1\right)^{\phi_{m+3}} \\
&= \left((x_2 y_1^{-\phi_{m+3}}\right)^{p_{m+4}} x_1^{\phi_{m+3}} \\
&= \left((x_2 y_1^{-\phi_{m+1}} (x_1^{\phi_{m+2}})^{-p_{m+3}}\right)^{p_{m+4}} x_1^{\phi_{m+2}} \\
&= (x_2 y_1^{-\phi_{m+3}})^{p_{m+4}-1} \circ x_2 \circ y_1^{-1} \circ x_1^{-p_{m+1}} \\
&\qquad \circ \left(x_1^{-1} \circ A_m^{p_m} \circ A_{m+2}^{-p_{m+2}} \circ y_1^{-1} \circ x_1^{-p_{m+1}}\right)^{p_{m+3}-1} \circ A_m^{p_m},
\end{aligned}$$

beginning with x_2 and ending with x_1^{-1}, moreover, the element that is cancelled out is $x_1^{\phi_{m+2}}$. Similarly,

$$\begin{aligned} y_1^{\phi_{m+4}} &= (x_2 y_1^{-\phi_{m+3}})^{p_{m+4}} y_1^{\phi_{m+3}} (y_1^{\phi_{m+3}} x_2^{-1})^{p_{m+4}} \\ &= (x_2 y_1^{-\phi_{m+3}})^{p_{m+4}-1} \circ x_2 \circ (y_1^{\phi_{m+3}} x_2^{-1})^{p_{m+4}} \\ &= A_{m+4}^{-(p_{m+4}-1)} \circ x_2 \circ A_{m+4}^{p_{m+4}}, \end{aligned}$$

beginning with x_2 and ending with x_2^{-1}, moreover, the element that is cancelled out is $y_1^{\phi_{m+3}}$.

This proves the lemma. $\square$

In the following lemmas we describe the reduced expressions of the elements $x_i^{\phi_j}$ and $y_i^{\phi_j}$.

Lemma 4.9. *Let $n \geqslant 2$, $K = K(m,n)$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, and*

$$\phi_K = \gamma_K^{p_K} \ldots \gamma_1^{p_1}.$$

Then for any i, $n \geqslant i \geqslant 2$, the following holds:

(1) *All automorphisms from Γ, except for $\gamma_{m+4(i-1)}, \gamma_{m+4i-2}, \gamma_{m+4i}$ fix x_i, and all automorphisms from Γ, except for $\gamma_{m+4(i-1)}$, γ_{m+4i-3}, γ_{m+4i-1}, γ_{m+4i} fix y_i. It follows that*

$$x_i^{\phi_K} = x_i^{\phi_{K-1}} = \ldots = x_i^{\phi_{m+4i}},$$

$$y_i^{\phi_K} = y_i^{\phi_{K-1}} = \ldots = y_i^{\phi_{m+4i}}.$$

(2) *Let $\tilde{y}_i = y_i^{\phi_{m+4i-1}}$. Then*

$$\tilde{y}_i = \left| \begin{array}{c} \tilde{y}_i \\ \hline {\scriptstyle x_i y_{i-1}^{-1} \qquad x_i y_i} \end{array} \right|$$

where (for $i = 1$) we assume that $y_0 = x_1^{-1}$ for $m = 0$, and $y_0 = z_m$ for $m \neq 0$;

(3) *Below we list the reduced forms of the leading terms of the corresponding automorphisms. Put $q_j = p_{m+4(i-1)+j}$ for $j = 0, \ldots, 4$. In the formulas below we assume that $y_0 = x_1^{-1}$ for $m = 0$, and $y_0 = z_m$ for $m \neq 0$.*

$$\begin{aligned} A_{m+4i-4} &= \left| \begin{array}{c} \tilde{y}_{i-1} \circ x_i^{-1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \qquad x_{i-1} y_{i-1} x_i^{-1}} \end{array} \right|, \\ SubC_3(A_{m+4i-4}) &= Sub_3(\tilde{y}_{i-1}) \\ &\qquad \cup \{x_{i-1} y_{i-1} x_i^{-1},\ y_{i-1} x_i^{-1} x_{i-1},\ x_i^{-1} x_{i-1} y_{i-2}^{-1}\}; \\ A_{m+4i-3} &= x_i; \\ A_{m+4i-2} &= \left| \begin{array}{c} A_{m+4i-4}^{-q_0} \\ \hline {\scriptstyle x_i y_{i-1}^{-1} \qquad y_{i-2} x_{i-1}^{-1}} \end{array} \right| x_i^{q_1} y_i, \\ SubC_3(A_{m+4i-2}) &= SubC_3(A_{m+4i-4}) \\ &\qquad \cup \{y_{i-2} x_{i-1}^{-1} x_i,\ x_{i-1}^{-1} x_i^2,\ x_i^2 y_i,\ x_i y_i x_i,\ y_i x_i y_{i-1}^{-1} x_i^3\}; \\ A_{m+4i-1} &= \left| \begin{array}{c|c} A_{m+4i-2}^{q_2-1} & A_{m+4i-4}^{-q_0} \\ \hline {\scriptstyle x_i y_{i-1}^{-1} \qquad x_i y_i} & {\scriptstyle x_i y_{i-1}^{-1} \qquad y_{i-2} x_{i-1}^{-1}} \end{array} \right| x_i^{q_1+1} y_i, \\ SubC_3(A_{m+4i-1}) &= SubC_3(A_{m+4i-2}). \end{aligned}$$

(4) *Below we list the reduced forms of elements* $x_i^{\phi_{m+4(i-1)+j}}, y_i^{\phi_{m+4(i-1)+j}}$ *for* $j = 0, \ldots, 4$. *Again, in the formulas below we assume that* $y_0 = x_1^{-1}$ *for* $m = 0$, *and* $y_0 = z_m$ *for* $m \neq 0$.

$$x_i^{\phi_{m+4i-4}} = A_{m+4i-4}^{-q_0} \circ x_i \circ A_{m+4i-4}^{q_0},$$

$$y_i^{\phi_{m+4i-4}} = A_{m+4i-4}^{-q_0} \circ y_i,$$

$$x_i^{\phi_{m+4i-3}} = x_i^{\phi_{m+4i-4}},$$

$$y_i^{\phi_{m+4i-3}} = A_{m+4i-4}^{-q_0} \circ x_i^{q_1} \circ y_i,$$

$$x_i^{\phi_{m+4i-2}} = \left|\frac{A_{m+4i-2}^{q_2}}{x_iy_{i-1}^{-1}\quad x_iy_i}\right|\frac{A_{m+4i-4}^{-q_0}}{x_iy_{i-1}^{-1}\quad y_{i-2}x_{i-1}^{-1}}\right| x_i \left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1}\quad y_{i-1}x_i^{-1}}\right|,$$

$$y_i^{\phi_{m+4i-2}} = y_i^{\phi_{m+4i-3}},$$

$$x_i^{\phi_{m+4i-1}} = x_i^{\phi_{m+4i-2}} =_{(when\ i=n)} x_i^{\phi_K},$$

$$Sub_3(x_i^{\phi_K}) =_{(when\ i=n)} SubC_3(A_{m+4i-2}) \ \cup\ SubC_3(A_{m+4i-4})^{\pm 1} \ \cup$$
$$\{y_{i-2}x_{i-1}^{-1}x_i,\ x_{i-1}^{-1}x_ix_{i-1},\ x_ix_{i-1}y_{i-2}^{-1}\};$$

$$y_i^{\phi_{m+4i-1}} = \tilde{y}_i =_{(when\ i=n)} y_i^{\phi_K} =$$
$$\left|\frac{A_{m+4i-4}^{-q_0}}{x_iy_{i-1}^{-1}\quad y_{i-2}x_{i-1}^{-1}}\right| \left(x_i^{q_1} y_i \left|\frac{A_{m+4i-2}^{q_2-1}}{x_iy_{i-1}^{-1}\quad x_iy_i}\right|\frac{A_{m+4i-4}^{-q_0}}{x_iy_{i-1}^{-1}\quad y_{i-2}x_{i-1}^{-1}}\right| x_i \right)^{q_3} x_i^{q_1} y_i,$$

$$Sub_3(\tilde{y}_i) = SubC_3(A_{m+4i-2}) \cup SubC_3(A_{m+4i-4})^{-1} \cup \{y_{i-2}x_{i-1}^{-1}x_i,\ x_{i-1}^{-1}x_i^2,$$
$$x_i^3,\ x_iy_ix_i,\ y_ix_iy_{i-1}^{-1},\ \ x_i^2y_i\}$$

$$x_i^{\phi_{m+4i}} =_{(when\ i\neq n)} x_i^{\phi_K} = \left|\frac{A_{m+4i}^{-q_4+1}}{x_{i+1}y_i^{-1}\quad y_{i-1}x_i^{-1}}\right| x_{i+1} \circ y_i^{-1} x_i^{-q_1} \circ$$
$$\circ \left(x_i^{-1} \left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1}\quad y_{i-1}x_i^{-1}}\right|\frac{A_{m+4i-2}^{-q_2+1}}{y_i^{-1}x_i^{-1}\quad y_{i-1}x_i^{-1}}\right| y_i^{-1} x_i^{-q_1} \right)^{q_3-1} \left|\frac{A_{m+4i-4}^{q_0}}{x_{i-1}y_{i-2}^{-1}\quad y_{i-1}x_i^{-1}}\right|,$$

$$Sub_3(x_i^{\phi_K}) = SubC_3(A_{m+4i})^{-1} \cup SubC_3(A_{m+4i-2})^{-1} \cup SubC_3(A_{m+4i-4})$$
$$\cup \{y_{i-1}x_i^{-1}x_{i+1},\ x_i^{-1}x_{i+1}y_i^{-1},\ x_{i+1}y_i^{-1}x_i^{-1},\ y_i^{-1}x_i^{-2},\ x_i^{-3},\ x_i^{-2}x_{i-1},$$
$$x_i^{-1}x_{i-1}y_{i-2}^{-1},\ y_{i-1}x_i^{-1}x_{i-1},\ y_{i-1}x_i^{-1}y_i^{-1},\ x_i^{-1}y_i^{-1}x_i^{-1}\};$$

$$y_i^{\phi_{m+4i}} \ = \ y_i^{\phi_K} \ = \left|\frac{A_{m+4i}^{-q_4+1}}{x_{i+1}y_i^{-1}\quad y_{i-1}x_i^{-1}}\right| x_{i+1} \left|\frac{\tilde{y}_i}{x_iy_{i-1}^{-1}\quad x_iy_i}\right| x_{i+1}^{-1} \left|\frac{A_{m+4i}^{q_4-1}}{x_iy_{i-1}^{-1}\quad y_ix_{i+1}^{-1}}\right|,$$

$$Sub_3(y_i^{\phi_K}) = SubC_3(A_{m+4i})^{\pm 1} \cup Sub_3(\tilde{y}_i) \cup \{y_{i-1}x_i^{-1}x_{i+1},$$
$$x_i^{-1}x_{i+1}x_i,\ x_{i+1}x_iy_{i-1}^{-1},\ x_iy_ix_{i+1}^{-1},\ y_ix_{i+1}^{-1}x_i,\ x_{i+1}^{-1}x_iy_{i-1}^{-1}\}.$$

PROOF. Statement (1) is obvious. We prove statement (2) by induction on $i \geqslant 2$. Notice that by Lemmas 4.7 and 4.8 $\tilde{y}_1 = y_1^{\phi_{m+3}}$ begins with x_1 and ends with y_1. Now let $i \geqslant 2$. Then

$$\begin{aligned}\tilde{y}_i &= y_i^{\phi_{m+4i-1}} \\ &= (x_i^{q_3} y_i)^{\phi_{m+4i-2}} \\ &= ((y_i^{q_2} x_i)^{q_3} y_i)^{\phi_{m+4i-3}} \\ &= \left(((x_i^{q_1} y_i)^{q_2} x_i)^{q_3}\, x_i^{q_1} y_i\right)^{\phi_{m+4i-4}}.\end{aligned}$$

Before we continue, and to avoid huge formulas, we compute separately $x_i^{\phi_{m+4i-4}}$ and $y_i^{\phi_{m+4i-4}}$:

$$\begin{aligned}x_i^{\phi_{m+4i-4}} &= \left(x_i^{(y_{i-1}x_i^{-1})^{q_0}}\right)^{\phi_{m+4(i-1)-1}} \\ &= x_i^{(\tilde{y}_{i-1}x_i^{-1})^{q_0}} \\ &= (x_i\tilde{y}_{i-1}^{-1})^{q_0} \circ x_i \circ (\tilde{y}_{i-1}x_i^{-1})^{q_0},\end{aligned}$$

by induction (by Lemmas 4.7 and 4.8 in the case $i = 2$) beginning with $x_i y_{i-1}^{-1}$ and ending with $y_{i-1}x_i^{-1}$.

$$\begin{aligned}y_i^{\phi_{m+4i-4}} &= \left((y_{i-1}x_i^{-1})^{-q_0} y_i\right)^{\phi_{m+4(i-1)-1}} \\ &= (\tilde{y}_{i-1}x_i^{-1})^{-q_0} y_i \\ &= (x_i \circ \tilde{y}_{i-1}^{-1})^{q_0} \circ y_i,\end{aligned}$$

beginning with $x_i y_{i-1}^{-1}$ and ending with $x_{i-1}^{-1} y_i$. It follows that

$$\begin{aligned}(x_i^{q_1} y_i)^{\phi_{m+4i-4}} &= (x_i\tilde{y}_{i-1}^{-1})^{q_0} \circ x_i^{q_1} \circ (\tilde{y}_{i-1}x_i^{-1})^{q_0} (x_i \circ \tilde{y}_{i-1}^{-1})^{q_0} \circ y_i \\ &= (x_i\tilde{y}_{i-1}^{-1})^{q_0} \circ x_i^{q_1} \circ y_i,\end{aligned}$$

beginning with $x_i y_{i-1}^{-1}$ and ending with $x_i y_i$. Now looking at the formula

$$\tilde{y}_i = \left(((x_i^{q_1} y_i)^{q_2} x_i)^{q_3}\, x_i^{q_1} y_i\right)^{\phi_{m+4i-4}}$$

it is obvious that $\tilde{y}_i$ begins with $x_i y_{i-1}^{-1}$ and ends with $x_i y_i$, as required.

Now we prove statements (3) and (4) simultaneously.

$A_{m+4i-4} = \operatorname{cycred}((y_{i-1}x_i^{-1})^{\phi_{m+4(i-1)-1}}) = \tilde{y}_{i-1} \circ x_i^{-1}$, beginning with x_{i-1} and ending with x_i^{-1}. As we have observed in proving (2)

$$x_i^{\phi_{m+4i-4}} = (x_i\tilde{y}_{i-1}^{-1})^{q_0} \circ x_i \circ (\tilde{y}_{i-1}x_i^{-1})^{q_0} = A_{m+4i-4}^{-q_0} \circ x_i \circ A_{m+4i-4}^{q_0},$$

beginning with x_i and ending with x_i^{-1}.

$$y_i^{\phi_{m+4i-4}} = (x_i \circ \tilde{y}_{i-1}^{-1})^{q_0} \circ y_i = A_{m+4i-4}^{-q_0} \circ y_i,$$

beginning with x_i and ending with y_i. Now

$A_{m+4i-3} = \operatorname{cycred}(x_i^{\phi_{m+4i-4}}) = x_i$, beginning with x_i and ending with x_i.

$$\begin{array}{rcl} x_i^{\phi_{m+4i-3}} & = & x_i^{\phi_{m+4i-4}}, \\ y_i^{\phi_{m+4i-3}} & = & (x_i^{q_1} y_i)^{\phi_{m+4i-4}} \\ & = & A_{m+4i-4}^{-q_0} \circ x_i^{q_1} \circ A_{m+4i-4}^{q_0} A_{m+4i-4}^{-q_0} \circ y_i \\ & = & A_{m+4i-4}^{-q_0} \circ x_i^{q_1} \circ y_i, \end{array}$$

beginning with x_i and ending with y_i. Now

$$\begin{array}{rcl} A_{m+4i-2} & = & y_i^{\phi_{m+4i-3}}, \\ x_i^{\phi_{m+4i-2}} & = & (y_i^{q_2} x_i)^{\phi_{m+4i-3}} \\ & = & A_{m+4i-2}^{q_2} \circ A_{m+4i-4}^{-q_0} \circ x_i \circ A_{m+4i-4}^{q_0}, \end{array}$$

beginning with x_i and ending with x_i^{-1}. It is also convenient to rewrite $x_i^{\phi_{m+4i-2}}$ (by rewriting the subword A_{m+4i-2}) to show its cyclically reduced form:

$$\begin{array}{rcl} x_i^{\phi_{m+4i-2}} & = & A_{m+4i-4}^{-q_0} \circ \left(x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i \right) \\ & & \circ A_{m+4i-4}^{q_0}. \\ y_i^{\phi_{m+4i-2}} & = & y_i^{\phi_{m+4i-3}}. \end{array}$$

Now we can write down the next set of formulas:

$$\begin{array}{rcl} A_{m+4i-1} & = & \operatorname{cycred}(y_i^{-\phi_{m+4i-3}} x_i^{\phi_{m+4i-2}} y_i^{\phi_{m+4i-3}}) \\ & = & \operatorname{cycred}(A_{m+4i-2}^{-1} A_{m+4i-2}^{q_2} A_{m+4i-4}^{-q_0} \\ & & \quad x_i A_{m+4i-4}^{q_0} A_{m+4i-2} \\ & = & A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i^{q_1+1} \circ y_i, \end{array}$$

beginning with x_i and ending with y_i,

$x_i^{\phi_{m+4i-1}} = x_i^{\phi_{m+4i-2}}$, $y_i^{\phi_{m+4i-1}} = \tilde{y}_i = (x_i^{q_3} y_i)^{\phi_{m+4i-2}} = (x_i^{\phi_{m+4i-2}})^{q_3} y_i^{\phi_{m+4i-2}} =$

substituting the cyclic decomposition of $x_i^{\phi_{m+4i-2}}$ from above one has

$$= A_{m+4i-4}^{-q_0} \circ \left(x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i \right)^{q_3} \circ x_i^{q_1} \circ y_i.$$

beginning with x_i and ending with y_i.

Finally

$$A_{m+4i} = (y_i x_{i+1}^{-1})^{\phi_{m+4i-1}} = \tilde{y}_i \circ x_{i+1}^{-1},$$

beginning with x_i and ending with x_{i+1}^{-1}.

$$\begin{array}{rcl} x_i^{\phi_{m+4i}} & = & \left((y_i x_{i+1}^{-1})^{-q_4} x_i \right)^{\phi_{m+4i-1}} \\ & = & (\tilde{y}_i x_{i+1}^{-1})^{-q_4} x_i^{\phi_{m+4i-1}} \\ & = & A_{m+4i}^{-q_4+1} x_{i+1} \tilde{y}_i^{-1} x_i^{\phi_{m+4i-1}} \\ & = & A_{m+4i}^{-q_4+1} \circ x_{i+1} \circ \left((x_i^{\phi_{m+4i-2}})^{q_3-1} y_i^{\phi_{m+4i-2}} \right)^{-1}. \end{array}$$

Observe that computations similar to that for $y_i^{\phi_{m+4i-1}}$ show that

$$\left((x_i^{\phi_{m+4i-2}})^{q_3-1} y_i^{\phi_{m+4i-2}}\right)^{-1} = \left(A_{m+4i-4}^{-q_0} \circ \left(x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i\right)^{q_3-1} \circ x_i^{q_1} \circ y_i\right)^{-1}.$$

Therefore

$$x_i^{\phi_{m+4i}} = A_{m+4i}^{-q_4+1} \circ x_{i+1} \circ \circ \left(A_{m+4i-4}^{-q_0} \circ \left(x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-q_0} \circ x_i\right)^{q_3-1} \circ x_i^{q_1} \circ y_i\right)^{-1},$$

beginning with x_{i+1} and ending with x_i^{-1}.

$$\begin{aligned} y_i^{\phi_{m+4i}} &= \left(y_i^{(y_i x_{i+1}^{-1})^{q_4}}\right)^{\phi_{m+4i-1}} \\ &= (x_{i+1}\tilde{y}_i^{-1})^{q_4} \tilde{y}_i (\tilde{y}_i x_{i+1}^{-1})^{q_4} \\ &= A_{m+4i}^{-q_4+1} \circ x_{i+1} \circ \tilde{y}_i \circ x_{i+1}^{-1} \circ A_{m+4i}^{q_4-1}, \end{aligned}$$

beginning with x_{i+1} and ending with x_{i+1}^{-1}. This finishes the proof of the lemma. □

LEMMA 4.10. *Let $m > 2, n = 0$, $K = K(m,n)$, $p = (p_1, \ldots, p_K)$ be a 3-large tuple, $\phi_K = \gamma_K^{p_K} \cdots \gamma_1^{p_1}$, and $X^{\pm\phi_K} = \{x^{\phi_K} \mid x \in X^{\pm 1}\}$. Then the following holds:*

(1) $$Sub_2(X^{\pm\phi_K}) = \left\{ \begin{array}{ll} c_j z_j,\ z_j^{-1} c_j & (1 \leqslant j \leqslant m), \\ z_j z_{j+1}^{-1} & (1 \leqslant j \leqslant m-1), \\ z_m x_1^{-1},\ z_m x_1 & (if\ m \neq 0, n \neq 0), \\ x_i^2,\ x_i y_i,\ y_i x_i & (1 \leqslant i \leqslant n), \\ x_{i+1} y_i^{-1},\ x_i^{-1} x_{i+1},\ x_{i+1} x_i & (1 \leqslant i \leqslant n-1) \end{array} \right\}^{\pm 1}$$

moreover, the word $z_j^{-1} c_j$, as well as $c_j z_j$, occurs only as a part of the subword $(z_j^{-1} c_j z_j)^{\pm 1}$ in x^{ϕ_K} $(x \in X^{\pm 1})$;

(2) $Sub_3(X^{\pm\phi_K}) =$

$$\left\{ \begin{array}{ll} z_j^{-1} c_j z_j, & (1 \leqslant j \leqslant m), \\ c_j z_j z_{j+1}^{-1},\ z_j z_{j+1}^{-1} c_{j+1}^{-1},\ z_j z_{j+1}^{-1} c_{j+1}, & (1 \leqslant j \leqslant m-1), \\ y_1 x_1^2, & (m = 0, n = 1), \\ x_2^{-1} x_1^2,\ x_2 x_1^2, & (m = 0, n \geqslant 2) \\ c_m^{-1} z_m x_1, & (m = 1, n \neq 0) \\ c_m z_m x_1^{-1},\ c_m z_m x_1,\ z_m x_1^{-1} z_m^{-1},\ z_m x_1^2,\ z_m x_1^{-1} y_1^{-1}, & (\ m \neq 0, n \neq 0), \\ z_m x_1^{-1} x_2,\ z_m x_1^{-1} x_2^{-1}, & (m \neq 0, n \geqslant 2), \\ c_1^{-1} z_1 z_2^{-1}, & (m \geqslant 2), \\ x_i^3,\ x_i^2 y_i,\ \ x_i y_i x_i, & (1 \leqslant i \leqslant n), \\ x_i^{-1} x_{i+1} x_i,\ y_i x_{i+1}^{-1} x_i,\ x_i y_i x_{i+1}^{-1}, & (1 \leqslant i \leqslant n-1), \\ x_{i-1}^{-1} x_i^2,\ y_i x_i y_{i-1}^{-1}, & (2 \leqslant i \leqslant n), \\ y_{i-2} x_{i-1}^{-1} x_i^{-1},\ y_{i-2} x_{i-1}^{-1} x_i & (3 \leqslant i \leqslant n). \end{array} \right\}^{\pm 1}.$$

(3) *for any 2-letter word* $uv \in Sub_2(X^{\pm\phi_K})$ *one has*

$$Sub_2(u^{\phi_K} v^{\phi_K}) \subseteq Sub_2(X^{\pm\phi_K}), \quad Sub_3(u^{\phi_K} v^{\phi_K}) \subseteq Sub_3(X^{\pm\phi_K}).$$

Proof. (1) and (2) follow by straightforward inspection of the reduced forms of elements x^{ϕ_K} in Lemmas 4.6, 4.7, 4.8, and 4.9.

To prove (3) it suffices for every word $uv \in Sub_2(X^{\pm\phi_K})$ to write down the product $u^{\phi_K} v^{\phi_K}$ (using formulas from the lemmas mentioned above), then make all possible cancellations and check whether 3-subwords of the resulting word all lie in $Sub_3(X^{\pm\phi_K})$. Now we do the checking one by one for all possible 2-words from $Sub_2(X^{\pm\phi_K})$.

1) For $uv \in \{c_j z_j,\ z_j^{-1} c_j\}$ the checking is obvious and we omit it.

2) Let $uv = z_j z_{j+1}^{-1}$. Then there are three cases to consider:

2.a) Let $j \leqslant m-2$, then

$$(z_j z_{j+1}^{-1})^{\phi_K} = \left|\begin{array}{c|c} z_j^{\phi_K} & z_{j+1}^{-\phi_K} \\ \hline * \quad c_{j+1} z_{j+1} & z_{j+2}^{-1} c_{j+2}^{-1} \quad * \end{array}\right|,$$

in this case there is no cancellation in $u^{\phi_K} v^{\phi_K}$. All 3-subwords of u^{ϕ_K} and v^{ϕ_K} are obviously in $Sub_3(X^{\pm\phi_K})$. So one needs only to check the new 3-subwords which arise "in between" u^{ϕ_K} and v^{ϕ_K} (below we will check only subwords of this type). These subwords are $c_{j+1} z_{j+1} z_{j+2}^{-1}$ and $z_{j+1} z_{j+2}^{-1} c_{j+2}^{-1}$ which both lie in $Sub_3(X^{\pm\phi_K})$.

2.b) Let $j = m-1$ and $n \neq 0$. Then

$$(z_{m-1} z_m^{-1})^{\phi_K} = \left|\begin{array}{c|c} z_{m-1}^{\phi_K} & z_m^{-\phi_K} \\ \hline * \quad c_m z_m & x_1 z_m^{-1} \quad * \end{array}\right|,$$

again, there is no cancellation in this case and the words "in between" are $c_m z_m x_1$ and $z_m x_1 z_m^{-1}$, which are in $Sub_3(X^{\pm\phi_K})$.

2.c) Let $j = m-1$ and $n = 0$. Then (below we put $\cdot$ at the place where the corresponding initial segment of u^{ϕ_K} and the corresponding terminal segment of v^{ϕ_K} meet)

$$\begin{aligned}
(z_{m-1} z_m^{-1})^{\phi_K} &= z_{m-1}^{\phi_K} \cdot z_m^{-\phi_K} \\
&= c_{m-1} z_{m-1} A_{m-4}^{p_{m-4}} c_m^{z_m} A_{m-1}^{p_{m-1}-1} \cdot A_{m-1}^{-p_{m-1}} z_m^{-1} \\
&\quad \text{(cancelling } A_{m-1}^{p_{m-1}-1} \text{ and substituting for} \\
&\quad\ A_{m-1}^{-1} \text{ its expression via the leading terms)} \\
&= c_{m-1} z_{m-1} A_{m-4}^{p_{m-4}} c_m^{z_m} \cdot (c_m^{-z_m} A_{m-4}^{-p_{m-4}} c_{m-1}^{-z_{m-1}} A_{m-4}^{p_{m-4}}) z_m^{-1} \\
&= z_{m-1} \left|\begin{array}{c} A_{m-4}^{p_{m-4}} \\ \hline z_{m-2}^{-1} \quad * \end{array}\right| z_m^{-1}.
\end{aligned}$$

Here $z_{m-1}^{\phi_K}$ is completely cancelled.

3.a) Let $n = 1$. Then

$$\begin{aligned}
(z_m x_1^{-1})^{\phi_K} &= c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m - 1} \cdot A_m^{-p_m} x_1^{-1} A_m^{p_m} A_{m+2}^{p_{m+2}} \\
&= c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} \cdot x_1 A_{m-1}^{-p_{m-1}} c_m^{-z_m} A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m} A_{m+2}^{p_{m+2}} \\
&= \left|\begin{array}{c} z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m} A_{m+2}^{p_{m+2}} \\ \hline z_m z_{m-1}^{-1} \qquad\qquad \end{array}\right|,
\end{aligned}$$

and $z_m^{\phi_K}$ is completely cancelled.

3.b) Let $n > 1$. Then

$$
\begin{aligned}
(z_m x_1^{-1})^{\phi_K} &= c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m-1} \\
&\quad A_m^{-p_m}(x_1^{-1} A_m^{p_m} A_{m+2}^{-p_{m+2}+1} y_1^{-1} x_1^{-p_{m+1}})^{-p_{m+3}+1} x_1^{p_{m+1}} y_1 x_2^{-1} A_{m+4}^{p_{m+4}-1} \\
&= c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{-1}(x_1^{-1} A_m^{p_m} A_{m+2}^{-p_{m+2}+1} y_1^{-1} x_1^{-p_{m+1}})^{-p_{m+3}+1} x_1^{p_{m+1}} y_1 x_2^{-1} A_{m+4}^{p_{m+4}-1} \\
&= c_m z_m A_{m-1}^{p_{m-1}} x_1^{-1} \cdot x_1 A_{m-1}^{-p_{m-1}} c_m^{-z_m} A_{m-1}^{p_{m-1}} (x_1^{-1} A_m^{p_m} A_{m+2}^{-p_{m+2}+1} y_1^{-1} x_1^{-p_{m+1}})^{-p_{m+3}+1} \\
&\quad x_1^{p_{m+1}} y_1 x_2^{-1} A_{m+4}^{p_{m+4}-1} \\
&= \left|\begin{array}{c} z_m A_{m-1}^{p_{m-1}} \\ \hline \scriptstyle z_m z_{m-1}^{-1} c_{m-1}^{-1} \end{array}\right|,
\end{aligned}
$$

and $z_m^{\phi_K}$ is completely cancelled.

4.a) Let $n = 1$. Then

$$
\begin{aligned}
(z_m x_1)^{\phi_K} &= z_m A_{m-1}^{p_{m-1}} x_1^{-1} A_m^{p_m-1} \cdot A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 A_m^{p_m} \\
&= \left|\begin{array}{c} z_m A_{m-1}^{p_{m-1}} \; ** \\ \hline \scriptstyle z_m z_{m-1}^{-1} c_{m-1}^{-1} \qquad * \end{array}\right|,
\end{aligned}
$$

and $z_m^{\phi_K}$ is completely cancelled.

4.b) Let $n > 1$. Then

$$
(z_m x_1)^{\phi_K} = \left|\begin{array}{c|c} z_m^{\phi_K} & x_1^{\phi_K} \\ \hline \scriptstyle * \quad z_m x_1^{-1} & \scriptstyle x_2 y_1^{-1} \quad * \end{array}\right|.
$$

5.a) Let $n = 1$. Then

$$
\begin{aligned}
x_1^{2\phi_K} &= A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 A_m^{p_m} \cdot A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 A_m^{p_m} \\
&= A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 A_m^{p_m} \cdot (A_m^{-p_m} x_1^{p_{m+1}} y_1) A_{m+2}^{p_{m+2}-1} A_m^{-p_m} x_1 A_m^{p_m} \\
&= A_{m+2}^{p_{m+2}} \left|\begin{array}{c} A_m^{-p_m} x_1 \\ \hline \scriptstyle * \quad z_m x_1 \end{array}\right| \cdot x_1^{p_{m+1}} y_1 ** .
\end{aligned}
$$

5.b) Let $n > 1$. Then

$$
x_1^{2\phi_K} = \left|\begin{array}{c|c} x_1^{\phi_K} & x_1^{\phi_K} \\ \hline \scriptstyle z_m x_1^{-1} & \scriptstyle x_2 y_1^{-1} \end{array}\right|.
$$

6.a) Let $1 < i < n$. Then

$$
\begin{aligned}
x_i^{2\phi_K} &= A_{m+4i}^{-q_4+1} x_{i+1} y_i^{-1} x_i^{-q_1} (x^{-1} A_{m+4i-4}^{q_0} A_{m+4i-2}^{-q_2+1} y_i^{-1} x_i^{-q_1})^{q_3-1} \\
&\quad \left|\begin{array}{c} A_{m+4i-4}^{q_0} \\ \hline \scriptstyle y_{i-1} x_i^{-1} \end{array}\right| \cdot \left|\begin{array}{c} A_{m+4i}^{-q_4+1} \\ \hline \scriptstyle x_{i+1} y_i^{-1} \end{array}\right| **.
\end{aligned}
$$

6.b)

$$
\begin{aligned}
x_n^{2\phi_K} &= A_{m+4n-2}^{q_2} A_{m+4n-4}^{-q_0} x_n A_{m+4n-4}^{q_0} \\
&\quad \cdot A_{m+4n-2}^{q_2} A_{m+4n-4}^{-q_0} x_n A_{m+4n-4}^{q_0} \\
&= A_{m+4n-2}^{q_2} A_{m+4n-4}^{-q_0} x_n A_{m+4n-4}^{q_0} \\
&\quad \cdot A_{m+4n-4}^{-q_0} x_n^{q_1} y_n A_{m+4n-2}^{q_2-1} A_{m+4n-4}^{-q_0} x_n A_{m+4n-4}^{q_0} \\
&= A_{m+4n-2}^{q_2} \left|\begin{array}{c} A_{m+4n-4}^{-q_0} x_n \\ \hline \scriptstyle x_{n-1} x_n \end{array}\right| \cdot x_n^{q_1} **.
\end{aligned}
$$

7.a) If $n = 1$. Then $(x_1 y_1)^{\phi_K} = A_{m+2}^{p_{m+2}} A_m^{-p_m} x_1 \cdot x_1^{p_{m+1}} **$.

7.b) If $n > 1$. Then $(x_1y_1)^{\phi_K} = \left|\begin{array}{c|c} x_1^{\phi_K} & y_1^{\phi_K} \\ \hline {\scriptstyle z_m x_1^{-1}} & {\scriptstyle x_2 y_1^{-1}} \end{array}\right|.$

7.c) If $1 < i < n$. Then

$$(x_iy_i)^{\phi_K} = \left|\begin{array}{c|c} x_i^{\phi_K} & y_i^{\phi_K} \\ \hline {\scriptstyle y_{i-1} x_i^{-1}} & {\scriptstyle x_{i+1} y_i^{-1}} \end{array}\right|.$$

7.d) $(x_ny_n)^{\phi_K} = \left|\begin{array}{c|c} x_n^{\phi_K} & y_n^{\phi_K} \\ \hline {\scriptstyle x_{n-1}^{-1} x_n} & {\scriptstyle x_n^2} \end{array}\right|.$

8a) If $n = 1$. Then

$$(y_1x_1)^{\phi_K} = \left|\begin{array}{c|c} y_1^{\phi_K} & x_1^{\phi_K} \\ \hline {\scriptstyle x_1 y_1} & {\scriptstyle x_1 z_m^{-1}} \end{array}\right|.$$

8.b) If $n > 1$. Then

$$\begin{aligned}
(y_1x_1)^{\phi_K} &= A_{m+4}^{-p_{m+4}+1} x_2 A_{m+4}^{p_{m+4}} \cdot A_{m+4}^{-p_{m+4}+1} x_2 y_1^{-1} x_1^{-p_{m+1}} \circ ** \\
&= A_{m+4}^{-p_{m+4}+1} x_2 A_{m+4} \cdot x_2 y_1^{-1} x_1^{-p_{m+1}} \circ ** \\
&= A_{m+4}^{-p_{m+4}+1} x_2 A_m^{-p_m} (x_1^{p_{m+1}} y_1 A_{m+2}^{p_{m+2}-1} A_m^{-p_m} x_1)^{p_{m+3}} x_1^{p_{m+1}} y_1 x_2^{-1} \\
&\qquad x_2 y_1^{-1} x_1^{-p_{m+1}} ()^{p_{m+3}-1} A_m^{p_m} \\
&= A_{m+4}^{-p_{m+4}+1} x_2 A_m^{-p_m} (x_1^{p_{m+1}} y_1 A_{m+2}^{p_{m+2}-1} \left|\begin{array}{c|c} A_m^{-p_m} x_1 & A_m \\ \hline {\scriptstyle z_m x_1} & {\scriptstyle z_m^{-1} c_m^{-1}} \end{array}\right|.
\end{aligned}$$

8.c) $(y_nx_n)^{\phi_K} = \left|\begin{array}{c|c} y_n^{\phi_K} & x_n^{\phi_K} \\ \hline {\scriptstyle x_n y_n} & {\scriptstyle x_n y_{n-1}} \end{array}\right|.$

9.a) If $n = 2$, then

$$(x_2y_1^{-1})^{\phi_K} = A_{m+6}^{q_2} A_{m+4}^{-1}.$$

9.b) If $n > 2$, $1 < i < n$. Then

$$\begin{aligned}
(x_iy_{i-1}^{-1})^{\phi_K} &= \left|\begin{array}{c} A_{m+4i}^{-q_4+1} \\ \hline {\scriptstyle x_{i+1} y_i^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| x_{i+1} \circ y_i^{-1} x_i^{-q_1} \\
&\qquad \circ \left(x_i^{-1} \left|\begin{array}{c|c} A_{m+4i-4}^{q_0} & A_{m+4i-2}^{-q_2+1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} & {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| y_i^{-1} x_i^{-q_1} \right)^{q_3-1} \\
&\qquad \left|\begin{array}{c} A_{m+4i-4}^{q_0} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| \cdot A_{m+4i-4}^{-q_0+1} \circ x_i \circ \tilde{y}_{i-1} \circ x_i^{-1} \left|\begin{array}{c} A_{m+4i-4}^{q_0-1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| \\
&= \left|\begin{array}{c} A_{m+4i}^{-q_4+1} \\ \hline {\scriptstyle x_{i+1} y_i^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| x_{i+1} \circ y_i^{-1} x_i^{-q_1} \\
&\qquad \circ \left(x_i^{-1} \left|\begin{array}{c|c} A_{m+4i-4}^{q_0} & A_{m+4i-2}^{-q_2+1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} & {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right| y_i^{-1} x_i^{-q_1} \right)^{q_3-1} \\
&\qquad x_i^{-1} \left|\begin{array}{c} A_{m+4i-4}^{q_0-1} \\ \hline {\scriptstyle x_{i-1} y_{i-2}^{-1} \quad y_{i-1} x_i^{-1}} \end{array}\right|.
\end{aligned}$$

9.c) $(x_ny_{n-1}^{-1})^{\phi_K} = \left|\begin{array}{c|c} A_{m+4n-2}^{q_2} & A_{m+4n-4} \\ \hline {\scriptstyle x_n y_n} & {\scriptstyle x_n y_{n-1}^{-1}} \end{array}\right|.$

10.a) Let $n = 2$, then

$$
\begin{aligned}
&(x_1^{-1}x_2)^{\phi_K}\\
= \;& A_m^{-p_m}(x_1^{p_{m+1}}y_1A_{m+2}^{p_{m+2}-1}A_m^{-p_m}x_1)^{p_{m+3}-1}x_1^{p_{m+1}}y_1x_2^{-1}A_{m+4}^{p_{m+4}-1}A_{m+6}^{p_{m+6}}A_{m+4}^{-p_{m+4}}x_2A_{m+4}^{p_{m+4}}\\
= \;& A_m^{-p_m}(x_1^{p_{m+1}}y_1A_{m+2}^{p_{m+2}-1}A_m^{-p_m}x_1)^{p_{m+3}-1}x_1^{p_{m+1}}y_1x_2^{-1}A_{m+4}^{p_{m+4}-1}\\
&\quad (A_{m+4}^{-p_{m+4}}x_2^{p_{m+5}}y_2)^{p_{m+6}}A_{m+4}^{-p_{m+4}}x_2A_{m+4}^{p_{m+4}}\\
= \;& A_m^{-p_m}(x_1^{p_{m+1}}y_1A_{m+2}^{p_{m+2}-1}A_m^{-p_m}x_1)^{p_{m+3}-1}x_1^{p_{m+1}}y_1x_2^{-1}\\
&\quad \cdot A_{m+4}^{-1}x_2^{p_m+5}y_2(A_{m+4}^{-p_{m+4}}x_2^{p_{m+5}}y_2)^{p_{m+6}-1}A_{m+4}^{-p_{m+4}}x_2A_{m+4}^{p_{m+4}}\\
= \;& \left|\begin{array}{c} A_m^{-p_m} \\ \hline \scriptstyle c_m z_m \end{array}\right|\\
&\quad \left|\begin{array}{c} x_1^{-1}A_m^{p_m} \\ \hline \scriptstyle x_1^{-1}z_m^{-1} \end{array}\right| A_{m+2}^{-p_{m+2}+1}y_1^{-1}x_1^{-p_{m+1}}A_m^{p_m}x_2^{p_{m+5}}y_2(A_{m+4}^{-p_{m+4}}x_2^{p_{m+5}}y_2)^{p_{m+6}-1}\\
&\qquad A_{m+4}^{-p_{m+4}}x_2A_{m+4}^{p_{m+4}}.
\end{aligned}
$$

10.b) If $1 < i < n-1$, then

$$(x_i^{-1}x_{i+1})^{\phi_K} = \left|\begin{array}{c|c} x_i^{-\phi_K} & x_{i+1}^{\phi_K} \\ \hline \scriptstyle y_i x_{i+1}^{-1} & \scriptstyle x_{i+2}y_{i+1}^{-1} \end{array}\right|.$$

10.c) Similarly to 10.a) we get

$$(x_{n-1}^{-1}x_n)^{\phi_K} = \left|\begin{array}{c} A_{2n+4n-8}^{-p_{m+4n-8}} \\ \hline \scriptstyle y_{n-3}x_{n-2}^{-1} \end{array}\right| \cdot \left|\begin{array}{c} x_{n-1}^{-1}A_{m+4n-8}^{p_{m+4n-8}} \\ \hline \scriptstyle x_{n-1}^{-1}x_{n-2} \end{array}\right| A_{m+4n-6}^{p_{m+4n-6}+1} **.$$

11.a) If $1 < i < n-1$, then

$$
\begin{aligned}
&(x_{i+1}x_i)^{\phi_K}\\
= \;& A_{m+4i+4}^{-q_8+1}x_{i+2}y_{i+1}^{-1}x_{i+1}^{-q_5}\left(x_{i+1}^{-1}A_{m+4i}^{q_4}A_{m_4i+2}^{-q_6+1}y_{i+1}^{-1}x_{i+1}^{-q_5}\right)^{q_7-1}A_{m+4i}^{q_4}\\
&\quad A_{m+4i}^{-q_4+1}x_{i+1}y_i^{-1}x_i^{-q_1}\left(x_i^{-1}A_{m+4i-4}^{q_0}A_{m_4i-2}^{-q_2+1}y_i^{-1}x_i^{-q_1}\right)^{q_3-1}A_{m+4i-4}^{q_0}\\
= \;& A_{m+4i+4}^{-q_8+1}x_{i+2}y_{i+1}^{-1}x_{i+1}^{-q_5}\left(x_{i+1}^{-1}A_{m+4i}^{q_4}A_{m_4i+2}^{-q_6+1}y_{i+1}^{-1}x_{i+1}^{-q_5}\right)^{q_7-1}A_{m+4i}\\
&\quad x_{i+1}y_i^{-1}x_i^{-q_1}\left(x_i^{-1}A_{m+4i-4}^{q_0}A_{m_4i-2}^{-q_2+1}y_i^{-1}x_i^{-q_1}\right)^{q_3-1}A_{m+4i-4}^{q_0}\\
= \;& A_{m+4i+4}^{-q_8+1}x_{i+2}y_{i+1}^{-1}x_{i+1}^{-q_5}\left(x_{i+1}^{-1}A_{m+4i}^{q_4}A_{m_4i+2}^{-q_6+1}y_{i+1}^{-1}x_{i+1}^{-q_5}\right)^{q_7-1}\\
&\quad A_{m+4i-4}^{-q_0}x_i^{q_1}y_iA_{m+4i-2}^{q_2-1}\left|\begin{array}{c|c} A_{m+4i-4}^{-q_0}x_i & A_{m+4i-4}^{q_0} \\ \hline \scriptstyle x_{i-1}^{-1}x_i & \scriptstyle x_{i-1}y_{i-2}^{-1} \end{array}\right|.
\end{aligned}
$$

11.b) If $n > 2$, then

$$(x_2x_1)^{\phi_K} = \; ** \left|\begin{array}{c|c} A_m^{-q_0}x_1 & A_m^{q_0} \\ \hline \scriptstyle z_m x_1 & \scriptstyle z_m^{-1}c_m^{-1} \end{array}\right|.$$

11.c)

$$
\begin{aligned}
(x_nx_{n-1})^{\phi_K} \;&=\; A_{m+4n-2}^{q_6}A_{m+4n-4}^{-q_4}x_nA_{m+4n-4}^{q_4}\cdot A_{m+4n-4}^{-q_4+1}x_ny_{n-1}^{-1}x_{n-1}^{-q_1}\\
&\qquad (x_{n-1}^{-1}A_{m+4n-8}^{q_0}A_{m+4n-6}^{-q_2+1}y_{n-1}^{-1}x_{n-1}^{-q_1})^{q_3-1}A_{m+4n-8}^{q_0}\\
&=\; ** \left|\begin{array}{c} A_{m+4n-8}^{-q_0}x_{n-1} \\ \hline \scriptstyle x_{n-2}^{-1}x_{n-1} \end{array}\right| \cdot \left|\begin{array}{c} A_{m+4n-8}^{q_0} \\ \hline \scriptstyle x_{n-2}y_{n-3}^{-1} \end{array}\right|.
\end{aligned}
$$

11.d) Similarly, if $n = 2$, then

$$(x_2x_1)^{\phi_K} = \ast\ast\left|\begin{array}{r|c} A_m^{-p_m}x_1 & A_m^{p_m} \\ \hline {\scriptstyle z_mx_1} & {\scriptstyle z_m^{-1}c_m^{-1}} \end{array}\right|.$$

This proves the lemma. $\square$

Lemma 4.11. *Let $m > 2$, $n = 0$, $K = K(m, 0)$. Let $p = (p_1, \ldots, p_K)$ be a 3-large tuple, $\phi_K = \gamma_K^{p_K} \ldots \gamma_1^{p_1}$, and $X^{\pm\phi_K} = \{x^{\phi_K} \mid x \in X^{\pm 1}\}$. Denote the element*

$$c_1^{z_1} \cdots c_m^{z_m} \in F(X \cup C_S)$$

by a new letter d. Then the following holds:

(1) *Every element from X^{ϕ_K} can be uniquely presented as a reduced product of elements and their inverses from the set*

$$X \cup \{c_1, \ldots, c_{m-1}, d\}$$

Moreover:

- *all elements $z_i^{\phi_K}, i \neq m$ have the form $z_i^{\phi_K} = c_iz_i\hat{z}_i$, where $\hat{z}_i$ is a words in the alphabet $\{c_1^{z_1}, \ldots, c_{m-1}^{z_{m-1}}, d\}$,*
- *$z_m^{\phi_K} = z_m\hat{z}_m$, where $\hat{z}_m$ is a word in the alphabet $\{c_1^{z_1}, \ldots, c_{m-1}^{z_{m-1}}, d\}$.*

When viewing elements from X^{ϕ_K} as elements in

$$F(X \cup \{c_1, \ldots, c_{m-1}, d\}),$$

the following holds:

(2) $$Sub_2(X^{\pm\phi_K}) = \left\{ \begin{array}{ll} c_jz_j & (1 \leqslant j \leqslant m), \\ z_j^{-1}c_j,\ z_jz_{j+1}^{-1} & (1 \leqslant j \leqslant m-1), \\ z_2d,\ dz_{m-1}^{-1} & \end{array} \right\}^{\pm 1}$$

Moreover:

- *the word $z_mz_{m-1}^{-1}$ occurs only in the beginning of $z_m^{\phi_K}$ as a part of the subword $z_mz_{m-1}^{-1}c_{m-1}^{-1}z_{m-1}$*
- *the words $z_2d,\ dz_{m-1}^{-1}$ occur only as parts of subwords*

$$(c_1^{z_1}c_2^{z_2})^2dz_{m-1}^{-1}c_{m-1}^{-1}z_{m-1}c_{m-1}$$

and $(c_1^{z_1}c_2^{z_2})^2d$.

(3) $$Sub_3(X^{\pm\phi_K}) = \left\{ \begin{array}{ll} z_j^{-1}c_jz_j,\ c_jz_jz_{j+1}^{-1},\ z_jz_{j+1}^{-1}c_{j+1}^{-1}, & (1 \leqslant j \leqslant m-1), \\ z_jz_{j+1}^{-1}c_{j+1} & (1 \leqslant j \leqslant m-2), \\ c_2z_2d,\ z_2dz_{m-1}^{-1},\ dz_{m-1}^{-1}c_{m-1}^{-1}, & \end{array} \right\}^{\pm 1}.$$

Proof. The lemma follows from Lemmas 4.6 and 4.10 by replacing all the products $c_1^{z_1} \ldots c_m^{z_m}$ in subwords of $X^{\pm\phi_K}$ by the letter d. $\square$

Notation 4.12. Let $m \neq 0$, and if $m = 1$, then $n \neq 1$; $K = K(m, n), p = (p_1, \ldots, p_K)$ be a 3-large tuple, and $\phi_K = \gamma_K^{p_K} \ldots \gamma_1^{p_1}$. Let $\mathcal{W}$ be the set of words in $F(X \cup C_S)$ with the following properties:

(1) If $v \in W$ then $Sub_3(v) \subseteq Sub_3(X^{\pm\phi_K}), Sub_2(v) \subseteq Sub_2(X^{\pm\phi_K})$;
(2) Every subword $x_i^{\pm 2}$ of $v \in W$ is contained in a subword $x_i^{\pm 3}$;
(3) Every subword $c_1^{\pm z_1}$ of $v \in W$ is contained in $(c_1^{z_1}c_2^{z_2})^{\pm 3}$ when $m \geq 2$ or in $(c_1^{z_1}x_1^{-1})^{\pm 3}$ when $m = 1$;
(4) Every subword $c_m^{\pm z_m}$ $(m \geq 3)$ is contained in $\left(\prod_{i=1}^m c_i^{z_i}\right)^{\pm 1}$.

(5) every subword $c_2^{\pm z_2}$ of $v \in W$ is contained either in $(c_1^{z_1}c_2^{z_2})^{\pm 3}$ or as the central occurrence of $c_2^{\pm z_2}$ in $(c_2^{-z_2}c_1^{-z_1})^3 c_2^{\pm z_2}(c_1^{z_1}c_2^{z_2})^3$ or in $(c_1 z_1 c_2^{z_2}(c_1^{z_1}c_2^{z_2})^3)^{\pm 1}$.

Definition 4.13. The following words are called *elementary periods*:

$$x_i, \quad c_1^{z_1}c_2^{z_2} \ (if\ m \geq 2), \quad c_1^{z_1}x_1^{-1} \ (if\ m = 1).$$

We call the squares (cubes) of elementary periods or their inverses elementary squares (cubes).

Notation 4.14. Denote by Y the following set of words

1) if $n \neq 0$ then $Y = \{x_i, y_i, c_j^{z_j} \mid i = 1, \ldots, n,\ j = 1, \ldots, m\}$.
2) if $n = 0$ then $Y = \{c_1^{z_1}, \ldots, c_{m-1}^{z_{m-1}},\ d\}$.

Notation 4.15. 1) Denote by $\mathcal{W}_\Gamma$ the set of all subwords of words in $\mathcal{W}$.

2) Denote by $\bar{\mathcal{W}}_\Gamma$ the set of all words $v \in \mathcal{W}_\Gamma$ that are freely reduced forms of products of elements from $Y^{\pm 1}$. In this case we say that these elements v are (group) words in the alphabet Y.

Lemma 4.16. *Let $v \in \mathcal{W}_\Gamma$. Then the following holds:*

(1) *If v begins and ends with an elementary square but not an elementary cube, then v belongs to the following set:*

$$\left\{\begin{array}{ll}
x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i x_{i-1} y_{i-2}^{-1} x_{i-2}^{-2},\ x_i^2 y_i x_i y_{i-1}^{-1} x_{i-1}^{-2}, & m \geqslant 2, n \neq 0\\
x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i^2,\ x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i y_{i-1}^{-1} x_{i-1}^{-2}, & \\
x_1^2 y_1 x_1 c_m^{\pm z_m} CD,\ D_1 C_1 c_m^{z_m} x_1 c_m^{-z_m} C_2 D_2, & \\
D_1^{-1} C_1 c_m^{z_m} x_1^{-1} x_2 x_1 c_m^{-z_m} C_2 D_2 & \\
D_1^{-1} C_1 c_m^{z_m} x_1^2,\ x_1^2 y_1 x_2^{-1} x_1 c_m^{-z_m} C_2 D_2, x_2^{-2} x_1 c_m^{-z_m} C_3 D_3, & \\
D_1^{-1} C D_2 & \\
 & \\
(c_1^{z_1}c_2^{z_2})^2 d c_{m-1}^{-z_{m-1}} \ldots (c_2^{-z_2}c_1^{-z_1})^2,\ z_m c_{m-1}^{-z_{m-1}} \ldots (c_2^{-z_2}c_1^{-z_1})^2, & m \geqslant 3, n = 0\\
D_1^{-1} C D_2 & \\
\prod_{i=m-1}^{1} c_i^{-z_i} (c_2^{-z_2}c_1^{-z_1})^2 & \\
 & \\
x_1^2 y_1 (x_1 c_1^{-z_1})^2,\ (c_1^{z_1} x_1^{-1})^2 x_2 (x_1 c_1^{-z_1})^2, & m = 1, n \geq 2\\
(x_1 c_1^{-z_1})^2 x_1^2,\ x_1^2 y_1 x_2^{-1} (x_1 c_1^{-z_1})^2, x_2^{-2} (x_1 c_1^{-z_1})^2, & \\
 & \\
x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i x_{i-1} y_{i-2}^{-1} x_{i-2}^{-2},\ x_i^2 y_i x_i y_{i-1}^{-1} x_{i-1}^{-2}, & m = 0,\ n > 1\\
x_{i-2}^2 y_{i-2} x_{i-1}^{-1} x_i y_{i-1}^{-1} x_{i-1}^{-2},\ x_1^2 y_1 x_2^{-1} x_1^2,\ x_1^{-2} x_2^{-1} x_1^2, & \\
x_1^2 y_1 x_1,\ x_2^2 y_2 x_2 &
\end{array}\right\}^{\pm 1},$$

where C_k is an arbitrary product of the type $\prod_j c_{i_j}^{\pm z_{i_j}}$ with $i_{j+1} = i_j \pm 1$,

$$D_k = (c_1^{z_1}c_2^{z_2})^{\pm 2}.$$

(2) *If v does not contain two elementary squares and begins (ends) with an elementary square, or contains no elementary squares, then v is a subword of one of the words above.*

Proof. Straightforward verification using the description of the set $Sub_3(X^{\pm\phi_K})$ from Lemma 4.10. $\square$

Definition 4.17. Let Y be an alphabet and E a set of words of length at least 2 in Y. We say that an occurrence of a word $w \in Y \cup E$ in a word v is *maximal* relative to E if it is not contained in any other (distinct from w) occurrence of a word from E in v. We say that a set of words W in the alphabet Y admits *Unique Factorization Property (UF)* with respect to E if every word $w \in W$ can be uniquely presented as a product

$$w = u_1 \ldots u_k$$

where u_i are maximal occurrences of words from $Y \cup E$. In this event the decomposition above is called *irreducible*.

Lemma 4.18. *Let Y be an alphabet and E a set of words of length at least 2 in Y. If a set of words W in the alphabet Y satisfies the following condition:*

- *if $w_1w_2w_3$ is a subword of a word from W and $w_1w_2, w_2w_3 \in E$ then $w_1w_2w_3 \in E$ then W admits (UF) with respect to E.*

Definition 4.19. Let Y be an alphabet, E a set of words of length at least 2 in Y and W a set of words in Y which admits (UF) relative to E. An automorphism $\phi \in AutF(Y)$ satisfies the Nielsen property with respect to W with exceptions E if for any word $z \in Y \cup E$ there exists a decomposition

$$z^\phi = L_z \circ M_z \circ R_z, \tag{8}$$

for some words $L_z, M_z, R_z \in F(Y)$ such that for any $u_1, u_2 \in Y \cup E$ with $u_1u_2 \in Sub(W) \smallsetminus E$ the words $L_{u_1} \circ M_{u_1}$ and $M_{u_2} \circ R_{u_2}$ occur as written in the reduced form of $u_1^\phi u_2^\phi$.

Lemma 4.20. *Let W be a set of words in the alphabet Y which admits (UF) with respect to a set of words E. If an automorphism $\phi \in AutF(Y)$ satisfies the Nielsen property with respect to W with exceptions E then for every $w \in W$ if $w = u_1 \ldots u_k$ is the irreducible decomposition of w then the words M_{u_i} occur as written (uncancelled) in the reduced form of w^ϕ.*

Proof. follows directly from definitions.

It is easy to show that if an automorphism ϕ satisfies the Nielsen property with respect to W and E as above, then for each word $z \in Y \cup E$ there exists a unique decomposition (8) with maximal length of M_z. In this event we call M_z the *middle* of z^ϕ (with respect to ϕ).

Set

$$\begin{aligned}
T(m,1) &= \left\{ c_s^{z_s}(s=1,\ldots,m), \prod_{i=1}^{m} c_i^{z_i} x_1 \prod_{i=m}^{1} c_i^{-z_i} \right\}^{\pm 1}, m \neq 1, \\
T(m,2) &= T(m,1) \\
&\quad \cup \left\{ \prod_{i=1}^{m} c_i^{z_i} x_1^{-1} x_2 x_1 \prod_{i=m}^{1} c_i^{-z_i},\ y_1 x_2^{-1} x_1 \prod_{i=m}^{1} c_i^{-z_i},\ \prod_{i=1}^{m} c_i^{z_i} x_1^{-1} y_1^{-1} \right\}^{\pm 1},
\end{aligned}$$

if $n \geqslant 3$ then put

$$T(m,n) = T(m,1) \cup \left\{ \prod_{i=1}^{m} c_i^{z_i} x_1^{-1} x_2^{-1},\ \prod_{i=1}^{m} c_i^{z_i} x_1^{-1} y_1^{-1} \right\}^{\pm 1} \cup T_1(m,n),$$

where

$$T_1(m,n) \;=\; \{y_{n-2}x_{n-1}^{-1}x_nx_{n-1}y_{n-2}^{-1},\ y_{r-2}x_{r-1}^{-1}x_r^{-1},\ y_{r-1}x_r^{-1}y_r^{-1},\\ y_{n-1}x_n^{-1}x_{n-1}y_{n-2}^{-1}\ \ (n > r \geqslant 2)\}^{\pm 1}.$$

Now, let $E(m,n) = \bigcup_{i\geqslant 2} Sub_i(T(m,n)) \cap \bar{\mathcal{W}}_\Gamma$.

LEMMA 4.21. *Let $m \neq 0, n \neq 0, K = K(m,n), p = (p_1,\ldots,p_K)$ be a 3-large tuple. Then the following holds:*

(1) *Let $w \in E(m,n)$, $v = v(w)$ be the leading variable of w, and $j = j(v)$ (see notations at the beginning of Section 4). Then the period $A_j^{p_j-1}$ occurs in w^{ϕ_K} and each occurrence of A_j^2 in w^{ϕ_j} is contained in some occurrence of $A_j^{p_j-1}$. Moreover, no square A_k^2 occurs in w for $k > j$.*

(2) *The automorphism ϕ_K satisfies the Nielsen property with respect to $\bar{\mathcal{W}}_\Gamma$ with exceptions $E(m,n)$. Moreover, the following conditions hold:*
   (a) $M_{x_j} = A_{m+4r-8}^{-p_{m+4r-8}+1}x_{r-1}$, *for $j \neq n$.*
   (b) $M_{x_n} = x_n^{q_1} \circ y_n \circ A_{m+4n-2}^{q_2-1} \circ A_{m+4n-4}^{-q_0} \circ x_n$
   (c) $M_{y_j} = y_j^{\phi_K}$, *for $j < n$.*
   (d) $M_{y_n} = \left(x_n^{q_1}y_n \left| \underset{x_ny_{n-1}^{-1} \quad x_ny_n}{A_{m+4n-2}^{q_2-1}} \right| \underset{x_ny_{n-1}^{-1} \quad y_{n-2}x_{n-1}^{-1}}{A_{m+4n-4}^{-q_0}} \right| x_n \right)^{q_3} x_n^{q_1}y_n.$
   (e) $M_w = w^{\phi_K}$ *for any $w \in E(m,n)$ except for the following words:*
      - $w_1 = y_{r-2}x_{r-1}^{-1}x_r^{-1}, 3 \le r \le n-1$, $w_2 = y_{r-1}x_r^{-1}y_r^{-1}, 2 \le r \le n-1$,
      - $w_3 = y_{n-2}x_{n-1}^{-1}x_n$, $w_4 = y_{n-2}x_{n-1}^{-1}x_ny_{n-1}^{-1}$, $w_5 = y_{n-2}x_{n-1}^{-1}x_nx_{n-1}^{-1}y_{n-2}^{-1}$, $w_6 = y_{n-2}x_{n-1}^{-1}x_nx_{n-1}$, $w_7 = y_{n-2}x_{n-1}^{-1}x_n^{-1}$, $w_8 = y_{n-1}x_n^{-1}$, $w_9 = x_{n-1}^{-1}x_n$, $w_{10} = x_{n-1}^{-1}x_ny_{n-1}^{-1}$, $w_{11} = x_{n-1}^{-1}x_nx_{n-1}y_{n-2}^{-1}$.
   (f) *The only letter that may occur in a word from $\mathcal{W}_\Gamma$ to the left of a subword $w \in \{w_1,\ldots,w_8\}$ ending with y_i ($i = r-1, r-2, n-1, n-2,\ i \ge 1$) is x_i the maximal number j such that L_w contains $A_j^{p_j-1}$ is $j = m+4i-2$, and $R_{w_1} = R_{w_2} = 1$,*

PROOF. We first exhibit the formulas for u^{ϕ_K}, where $u \in \bigcup_{i\geqslant 2} Sub_i(T_1(m,n))$.

(1.a) Let $i < n$. Then

$$
\begin{aligned}
(x_i y_{i-1}^{-1})^{\phi_{m+4i}} &= (x_i y_{i-1}^{-1})^{\phi_K} \\
&= \left| \underset{x_{i+1}y_i^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i}^{-q_4+1}}} \right| x_{i+1} \circ y_i^{-1} x_i^{-q_1} \\
&\qquad \circ \left(x_i^{-1} \left| \underset{x_{i-1}y_{i-2}^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i-4}^{q_0}}} \right| \underset{x_{i-1}y_{i-2}^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i-2}^{-q_2+1}}} \right| y_i^{-1} x_i^{-q_1} \right)^{q_3-1} \\
&\qquad \left| \underset{x_{i-1}y_{i-2}^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i-4}^{q_0}}} \right| \\
&\qquad \cdot A_{m+4i-4}^{-q_0+1} \circ x_i \circ \tilde{y}_{i-1} \circ x_i^{-1} \left| \underset{x_{i-1}y_{i-2}^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i-4}^{q_0-1}}} \right| \\
&= \left| \underset{x_{i+1}y_i^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i}^{-q_4+1}}} \right| x_{i+1} \circ y_i^{-1} x_i^{-q_1} \\
&\qquad \circ \left(x_i^{-1} \left| \underset{x_{i-1}y_{i-2}^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i-4}^{q_0}}} \right| \underset{x_{i-1}y_{i-2}^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i-2}^{-q_2+1}}} \right| y_i^{-1} x_i^{-q_1} \right)^{q_3-1} \\
&\qquad \cdot x_i^{-1} \left| \underset{x_{i-1}y_{i-2}^{-1} \qquad y_{i-1}x_i^{-1}}{\underline{A_{m+4i-4}^{q_0-1}}} \right| .
\end{aligned}
$$

(1.b) Let $i = n$. Then

$$
\begin{aligned}
(x_n y_{n-1}^{-1})^{\phi_{m+4n-1}} &= (x_n y_{n-1}^{-1})^{\phi_K} \\
&= \left| \underset{x_n y_{n-1}^{-1} \qquad x_n y_n}{\underline{A_{m+4n-2}^{q_2}}} \right| \underset{x_n y_{n-1}^{-1} \qquad y_{n-2}x_{n-1}^{-1}}{\underline{A_{m+4n-4}^{-1}}} \right| .
\end{aligned}
$$

Here $y_{n-1}^{-\phi_K}$ is completely cancelled.

(2.a) Let $i < n-1$. Then

$$
\begin{aligned}
(x_{i+1} x_i y_{i-1}^{-1})^{\phi_K} &= (x_{i+1} x_i y_{i-1}^{-1})^{\phi_{m+4i+4}} \\
&= A_{m+4i+4}^{-q_8+1} \circ x_{i+2} \circ y_{i+1}^{-1} \circ x_{i+1}^{-q_5} \\
&\qquad \circ \left(x_{i+1}^{-1} \circ A_{m+4i}^{q_4} \circ A_{m+4i+2}^{-q_6+1} \circ y_{i+1}^{-1} x_{i+1}^{-q_5} \right)^{q_7-1} A_{m+4i-4}^{-q_0} \\
&\qquad \circ x_i^{q_1} y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-1}.
\end{aligned}
$$

Here $(x_i y_{i-1}^{-1})^{\phi_{m+4i+4}}$ was completely cancelled.

(2.b) Similarly, $(x_i y_{i-1}^{-1})^{\phi_{m+4i+3}}$ is completely cancelled in $(x_{i+1} x_i y_{i-1}^{-1})^{\phi_{m+4i+3}}$ and

$$
(x_{i+1} x_i y_{i-1}^{-1})^{\phi_{m+4i+3}} = A_{m+4i+2}^{q_6} \circ A_{m+4i}^{-q_4} \circ x_{i+1} \circ A_{m+4i-4}^{-q_0} A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-1}.
$$

(2.c)

$$
\begin{aligned}
(x_n^{-1} x_{n-1} y_{n-2}^{-1})^{\phi_{m+4n-1}} &= A_{m+4n-4}^{-q_4} \circ x_n^{-1} \circ A_{m+4n-4}^{q_4} \circ A_{m+4n-2}^{-q_6+1} \circ y_n^{-1} \circ x_n^{-q_5} \\
&\qquad \circ A_{m+4n-8}^{-q_0} \circ x_{n-1}^{q_1} \circ y_{n-1} \circ A_{m+4n-6}^{q_2-1} \circ A_{m+4n-8}^{-1},
\end{aligned}
$$

and $(x_{n-1} y_{n-2}^{-1})^{\phi_{m+4n-1}}$ is completely cancelled.

(3.a)

$$(y_i x_i y_{i-1}^{-1})^{\phi_{m+4i}} = A_{m+4i}^{-q_4+1} \circ x_{i+1} \circ A_{m+4i-4}^{-q_0} \circ x_i^{q_1} \circ y_i \circ A_{m+4i-2}^{q_2-1} \circ A_{m+4i-4}^{-1},$$

and $(x_i y_{i-1}^{-1})^{\phi_{m+4i}}$ is completely cancelled.

(3.b) $(y_n x_n y_{n-1}^{-1})^{\phi_K} = y_n^{\phi_K} \circ (x_n y_{n-1}^{-1})^{\phi_K}$.

(3.c)

$$\begin{aligned}(y_{n-1} x_n^{-1} x_{n-1} y_{n-2}^{-1})^{\phi_K} \quad &= \quad A_{m+4n-4} \circ A_{m+4n-2}^{-q_6+1} \circ y_n^{-1} \circ x_n^{-q_5} \\ &\qquad \circ A_{m+4n-8}^{-q_0} \circ x_{n-1}^{q_1} \circ y_{n-1} \circ A_{m+4n-6}^{q_2-1} \circ A_{m+4n-8}^{-1},\end{aligned}$$

and $y_{n-1}^{\phi_K}$ and $(x_{n-1} y_{n-2}^{-1})^{\phi_K}$ are completely cancelled.

(4.a) Let $n \geqslant 2$.

$$\begin{aligned}(x_1 c_m^{-z_m})^{\phi_{m+4i}} \quad &= \quad (x_1 c_m^{-z_m})^{\phi_K} \\ &= \quad \left|\underset{x_1 y_1^{-1} \qquad c_m^{z_m} x_1^{-1}}{A_{m+4}^{-q_4+1}}\right| x_2 \circ y_1^{-1} x_1^{-q_1} \\ &\qquad \circ \left(x_1^{-1} \circ A_m^{q_0} \circ A_{m+2}^{-q_2+1} \circ y_1^{-1} \circ x_1^{-q_1}\right)^{q_3-1} \\ &\qquad \circ A_m^{q_0} \cdot A_m^{-q_0} \circ x_1^{-1} \circ A_m^{q_0-1} \\ &= \quad A_{m+4}^{-q_4+1} \circ x_2 \circ y_1^{-1} \circ x_1^{-q_1} \\ &\qquad \circ \left(x_1^{-1} \circ A_m^{q_0} \circ A_{m+2}^{-q_2+1} \circ y_1^{-1} \circ x_1^{-q_1}\right)^{q_3-1} \circ x_1^{-1} \circ A_m^{q_0-1}.\end{aligned}$$

Let $n = 1$.

$$\begin{aligned}(x_1 z_m^{-c_m})^{\phi_K} \quad &= \quad A_m^{-p_m} \circ x_1^{p_{m+1}} \circ y_1 \circ A_{m+2}^{p_{m+2}-1} \circ A_m^{-1}, \\ (y_1 x_1 z_m^{-c_m})^{\phi_K} \quad &= \quad y_1^{\phi_K} \circ (x_1 z_m^{-c_m})^{\phi_K}.\end{aligned}$$

(4.b) $(x_1 c_m^{-z_m})^{\phi_K}$ is completely cancelled in $x_2^{\phi_K}$ and for $n > 2$:

$$\begin{aligned}(x_2 x_1 c_m^{-z_m})^{\phi_K} \quad &= \quad A_{m+8}^{-q_8+1} \circ x_3 \circ y_2^{-1} \circ x_3^{-q_5} \\ &\qquad \circ \left(x_3^{-1} \circ A_{m+4}^{q_4} \circ A_{m+6}^{-q_6+1} \circ y_2^{-1} \circ x_3^{-q_5}\right)^{q_7-1} \\ &\qquad \circ A_m^{-q_0} \circ x_1^{q_1} \circ y_1 \circ A_{m+2}^{q_2-1} \circ A_m^{-1}\end{aligned}$$

and for $n = 2$:

$$(x_2 x_1 c_m^{-z_m})^{\phi_K} = A_{m+6}^{q_6} \circ A_{m+4}^{-q_4} \circ x_i \circ A_m^{-q_0} \circ x_1^{q_1} \circ y_1 \circ A_{m+2}^{q_2-1} \circ A_m^{-1}.$$

(4.c) The cancellation between $(x_2 x_1 c_m^{-z_m})^{\phi_K}$ and $c_{m-1}^{-z_{m-1}}$ is the same as the cancellation between A_m^{-1} and $c_{m-1}^{-z_{m-1}^{\phi_K}}$, namely,

$$\begin{aligned}A_m^{-1} c_{m-1}^{-z_{m-1}^{\phi_K}} \quad &= \quad \left(x_1 \circ A_{m-1}^{-p_{m-1}} \circ c_m^{-z_m} \circ A_{m-1}^{p_{m-1}}\right) \\ &\qquad \left(A_{m-1}^{-p_{m-1}+1} \circ c_m^{-z_m} \circ A_{m-4}^{-p_{m-4}} \circ c_{m-1}^{-z_{m-1}} \circ A_{m-4}^{p_{m-4}} \circ c_m^{z_m} \circ A_{m-1}^{p_{m-1}-1}\right) \\ &= \quad x_1 A_{m-1}^{-1},\end{aligned}$$

and $c_{m-1}^{-z_{m-1}^{\phi_K}}$ is completely cancelled.

(4.d) The cancellations between $(x_2x_1c_m^{-z_m})^{\phi_K}$ (or between $(y_1x_1c_m^{-z_m})^{\phi_K}$) and $\prod_{i=m-1}^{1} c_i^{-z_i^{\phi_K}}$ are the same as the cancellations between A_m^{-1} and $\prod_{i=m-1}^{1} c_i^{-z_i^{\phi_K}}$ namely, the product $\prod_{i=m-1}^{1} c_i^{-z_i^{\phi_K}}$ is completely cancelled and

$$A_m^{-1}\prod_{i=m-1}^{1} c_i^{-z_i^{\phi_K}} = x_1\prod_{i=m}^{1} c_i^{-z_i}.$$

Similarly one can write expressions for u^{ϕ_K} for all $u \in E(m,n)$. The first statement of the lemma now follows from these formulas.

Let us verify the second statement. Suppose $w \in E(m,n)$ is a maximal subword from $E(m,n)$ of a word u from $\mathcal{W}_\Gamma$. If w is a subword of a word in $T(m,n)$, then either u begins with w or w is the leftmost subword of a word in $T(m,n)$. All the words in $T_1(m,n)$ begin with some y_j, therefore the only possible letters in u in front of w are x_j^2.

We have $x_j^{\phi_K}x_j^{\phi_K}w^{\phi_K} = x_j^{\phi_K} \circ x_j^{\phi_K} \circ w^{\phi_K}$ if w is a two-letter word, and $x_j^{\phi_K}x_j^{\phi_K}w^{\phi_K} = x_j^{\phi_K} \circ x_j^{\phi_K}w^{\phi_K}$ if w is more than a two-letter word. In this last case there are some cancellations between $x_j^{\phi_K}$ and w^{ϕ_K}, and the middle of x_j is the non-cancelled part of x_j because x_j as a letter not belonging to $E(m,n)$ appears only in x_j^n.

We still have to consider all letters that can appear to the right of w, if w is the end of some word in $T_1(m,n)$ or $w = y_{n-1}x_n^{-1}x_{n-1}$, $w = y_{n-1}x_n^{-1}$. There are the following possibilities:

(i) w is an end of $y_{n-2}x_{n-1}^{-1}x_nx_{n-1}y_{n-2}^{-1}$;
(ii) w is an end of $y_{r-2}x_{r-1}^{-1}x_r^{-1}, r < i$;
(iii) w is an end of $y_{n-2}x_{n-1}^{-1}y_{n-1}^{-1}$.

Situation (i) is equivalent to the situation when w^{-1} is the beginning of the word $y_{n-2}x_{n-1}^{-1}x_nx_{n-1}y_{n-2}^{-1}$, we have considered this case already. In the situation (ii) the only possible word to the right of w will be left end of $x_{r-1}y_{r-2}^{-1}x_{r-2}^{-2}$ and $w^{\phi_K}x_{r-1}^{\phi_K}y_{r-2}^{-\phi_K}x_{r-2}^{-2\phi_K} = w^{\phi_K} \circ x_{r-1}^{\phi_K}y_{r-2}^{-\phi_K} \circ x_{r-2}^{-2\phi_K}$, and $w^{\phi_K}x_{r-1}^{\phi_K} = w^{\phi_K} \circ x_{r-1}^{\phi_K}$. In the situation (iii) the first two letters to the right of w are $x_{n-1}x_{n-1}$, and $w^{\phi_K}x_{n-1}^{\phi_K} = w^{\phi_K} \circ x_{n-1}^{\phi_K}$.

There is no cancellation in the words $(c_j^{z_j})^{\phi_K} \circ (c_{j+1}^{\pm z_{j+1}})^{\phi_K}, (c_m^{z_m})^{\phi_K} \circ x_1^{\pm\phi_K}$, $x_1^{\phi_K} \circ x_1^{\phi_K}$. For all the other occurrences of x_i in the words from $\mathcal{W}_\Gamma$, namely for occurrences in x_i^n, $x_i^2y_i$, we have $(x_i^2y_i)^{\phi_K} = x_i^{\phi_K} \circ x_i^{\phi_K} \circ y_i^{\phi_k}$ for $i < n$.

In the case $n = i$, the bold subword of the word

$$x_n^{\phi_K} = A_{m+4n-4}^{-q_0} \circ \left(\mathbf{x_n^{q_1}} \circ \mathbf{y_n} \circ \mathbf{A_{m+4n-2}^{q_2-1}} \circ \mathbf{A_{m+4n-4}^{-q_0}} \circ \mathbf{x_n}\right) \circ A_{m+4n-4}^{q_0}$$

is M_{x_n} for ϕ_K, and the bold subword in the word

$$y_n^{\phi_K} = \left|\begin{array}{c} A_{m+4n-4}^{-q_0} \\ \hline x_ny_{n-1}^{-1} \quad y_{n-2}x_{n-1}^{-1} \end{array}\right| \left(\mathbf{x_n^{q_1}y_n} \left|\begin{array}{c|c} \mathbf{A_{m+4n-2}^{q_2-1}} & \mathbf{A_{m+4n-4}^{-q_0}} \\ \hline \mathbf{x_ny_{n-1}^{-1} \quad x_ny_n} & \mathbf{x_ny_{n-1}^{-1} \quad y_{n-2}x_{n-1}^{-1}} \end{array}\right| \mathbf{x_n}\right)^{\mathbf{q_3}} \mathbf{x_n^{q_1}y_n},$$

is M_{y_n} for ϕ_K.

□

Corollary 4.22. *Let $m \neq 0, n \neq 0, K = K(m,n), p = (p_1, \ldots, p_K)$ be a 3-large tuple, $L = Kl$. Then for any $u \in X \cup E(m,n)$ the element M_u with respect to ϕ_L contains A_j^q for some $j > L - K$ and $q > p_j - 3$.*

PROOF. This follows from the formulas for M_u with respect to ϕ_K in the lemma above. □

NOTATION 4.23. 1) Denote by $\mathcal{W}_{\Gamma,L}$ the least set of words in the alphabet Y that contains $\bar{\mathcal{W}}_\Gamma$, is closed under taking subwords, and is ϕ_K-invariant.

2) Let $\bar{\mathcal{W}}_{\Gamma,L}$ be union of $\mathcal{W}_{\Gamma,L}$ and the set of all initial subwords of $z_i^{\phi_{Kj}}$ which are of the form $c_i^j \circ z_i \circ w$, where $w \in \mathcal{W}_{\Gamma,L}$.

REMARK 4.24. The set $\bar{\mathcal{W}}_{\Gamma,L}$ is ϕ_K-invariant.

PROOF. Indeed, if $c_i^j z_i w \in \bar{\mathcal{W}}_{\Gamma,L}$, then $c_i^{c_i^j z_i w} = w^{-1} \circ c_i^{z_i} \circ w \in \mathcal{W}_{\Gamma,L}$ and $c_i^{(c_i^j z_i w)^{\phi_K}} = w^{-\phi_K} \circ c_i^{z_i^{\phi_K}} \circ w^{\phi_K} \in \mathcal{W}_{\Gamma,L}$, therefore $c_i^{j+1} z_i^{\phi_K} \circ w^{\phi_K} \in \bar{\mathcal{W}}_{\Gamma,L}$. □

NOTATION 4.25. Denote by Exc the following set of words in the alphabet Y.

$$Exc = \{c_1^{-z_1} c_i^{-z_i} c_{i-1}^{-z_{i-1}},\ c_1^{-z_1} x_1 c_m^{-z_m},\ c_1^{-z_1} x_j y_{j-1}^{-1}\}.$$

LEMMA 4.26. *The following holds:*

(1) $Sub_{3,Y}(\mathcal{W}_{\Gamma,L}) = Sub_{3,Y}(X^{\pm\phi_K}) \cup Exc.$

(2) *Let $v \in \mathcal{W}_{\Gamma,L}$ be a word that begins and ends with an elementary square and does not contain any elementary cubes. Then either $v \in \bar{\mathcal{W}}_\Gamma$ or $v = v_1 v_2$ where $v_1, v_2 \in \bar{\mathcal{W}}_\Gamma$ and these words are exhibited below:*

(a) *for $m > 2,\ n \geq 2$,*

$$v_1 \in \{v_{11} = (c_1^{z_1} c_2^{z_2})^2 \prod_{i=3}^{m} c_i^{z_i} x_1 x_2 x_1 \prod_{i=m}^{1} c_i^{-z_i},\ v_{12} = x_1^2 y_1 x_1 \prod_{i=m}^{1} c_i^{-z_i}\},$$

$$v_2 \in \{v_{2i} = c_i^{-z_i} \dots c_3^{-z_3} (c_2^{-z_2} c_1^{-z_1})^2, u_{2,1} = x_1 c_m^{-z_m} \dots c_3^{-z_3} (c_2^{-z_1} c_1^{-z_1})^2,$$
$$u_{2,j} = x_j y_{j-1}^{-1} x_{j-1}^2\};$$

(b) *for $m = 2,\ n \geq 2$,*

$$v_1 \in \{v_{11} = (c_1^{z_1} c_2^{z_2})^2 x_1 x_2 x_1 \prod_{i=m}^{1} c_i^{-z_i},\ v_{12} = x_1^2 y_1 x_1 \prod_{i=m}^{1} c_i^{-z_i}\},$$

$$v_2 \in \{u_{2,1} = x_1 (c_2^{-z_1} c_1^{-z_1})^2,\ u_{2,j} = x_j y_{j-1}^{-1} x_{j-1}^2\};$$

(c) *for $m > 2,\ n = 1$, $v_1 = x_1^2 y_1 x_1 \prod_{i=m}^{1} c_i^{-z_i}$,*

$$v_2 \in \{v_{2i} = c_i^{-z_i} \dots c_3^{-z_3} (c_2^{-z_2} c_1^{-z_1})^2,\ u_{2,1} = x_1 c_m^{-z_m} \dots c_3^{-z_3} (c_2^{-z_1} c_1^{-z_1})^2\};$$

(d) *for $m = 2,\ n = 1$, $v_1 = x_1^2 y_1 x_1 \prod_{i=m}^{1} c_i^{-z_i}$, $v_2 = x_1 (c_2^{-z_1} c_1^{-z_1})^2$;*

(e) *for $m = 1,\ n \geq 2$,*

$$v_1 \in \{v_{11} = (c_1^{z_1} x_1^{-1})^2 x_2 x_1 c_1^{-z_1},\ v_{12} = x_1^2 y_1 x_1 c_1^{-z_1}\},\ v_2 = x_j y_{j-1}^{-1} x_{j-1}^2.$$

PROOF. Let $T = Kl$. We will consider only the case $m \geq 2,\ n \geq 2$. We will prove the statement of the lemma by induction on l. If $l = 1$, then $T = K$ and the statement is true. Suppose now that

$$Sub_{3,Y}(\bar{\mathcal{W}}_\Gamma^{\phi_{T-K}}) = Sub_{3,Y}(\bar{\mathcal{W}}_\Gamma) \cup Exc.$$

Formulas in the beginning of the proof of Lemma 4.21 show that

$$Sub_{3,Y}(E(m,n)^{\pm\phi_K}) \subseteq Sub_{3,Y}(\bar{\mathcal{W}}_\Gamma).$$

By the second statement the automorphism ϕ_K satisfies the Nielsen property with exceptions $E(m,n)$. Let us verify that new 3-letter subwords do not occur "between" u^{ϕ_K} for $u \in T_1(m,n)$ and the power of the corresponding x_i to the left and right of it. All the cases are similar to the following:

$$(x_n x_{n-1} y_{n-2}^{-1})^{\phi_K} \cdot x_{n-2}^{\phi_K} \cdots \left| \begin{array}{c} A_{m+4n-10}^{-q+1} \\ \hline * \quad y_{n-3}x_{n-2}^{-1} \end{array} \right| \cdot x_{n-1}^{-1} \left| \begin{array}{c} A_{m+4n-8}^{q_0-1} \\ \hline x_{n-2} \quad\quad * \end{array} \right| \cdot$$

Words

$$(v_1 v_2)^{\phi_K}$$

produce the subwords from Exc. Indeed, $[(x_2 x_1 \prod_{i=m}^{1} c_i^{-z_i})]^{\phi_{Kj}}$ ends with v_{12} and $v_{12}^{\phi_K}$ ends with v_{12}. Similarly, $v_{2,j}^{\phi_K}$ begins with $v_{2,j+1}$ for $j < m$ and with $u_{2,1}$ for $j = m$. And $u_{2,j}^{\phi_K}$ begins with $u_{2,j+1}$ for $j < n$ and with $u_{2,j}$ for $j = n$.

This and the second part of Lemma 4.10 finish the proof. $\square$

Let $W \in G[X]$. We say that a word $U \in G[X]$ *occurs* in W if $W = W_1 \circ U \circ W_2$ for some $W_1, W_2 \in G[X]$. An occurrence of U^q in W is called *maximal* with respect to a property P of words if U^q is not a part of any occurrence of U^r with $q < r$ and which satisfies P. We say that an occurrence of U^q in W is *stable* if $q \geqslant 1$ and $W = W_1 \circ UU^qU \circ W_2$ (it follows that U is cyclically reduced). Maximal stable occurrences U^q will play an important part in what follows. If $(U^{-1})^q$ is a stable occurrence of U^{-1} in W then, sometimes, we say that U^{-q} is a stable occurrence of U in W. Two given occurrences U^q and U^p in a word W are *disjoint* if they do not have a common letter as subwords of W. Observe that if integers p and q have different signs then any two occurrences of A^q and A^p are disjoint. Also, any two different maximal stable occurrences of powers of U are disjoint. To explain the main property of stable occurrences of powers of U, we need the following definition. We say that a given occurrence of U^q *occurs correctly* in a given occurrence of U^p if $|q| \leqslant |p|$ and for these occurrences U^q and U^p one has $U^p = U^{p_1} \circ U^q \circ U^{p_1}$. We say, that two given non-disjoint occurrences of U^q, U^p *overlap correctly* in W if their common subword occurs correctly in each of them.

A cyclically reduced word A from $G[X]$ which is not a proper power and does not belong to G is called *a period*.

LEMMA 4.27. *Let A be a period in $G[X]$ and $W \in G[X]$. Then any two stable occurrences of powers of A in W are either disjoint or they overlap correctly.*

PROOF. Let A^q, A^p $(q \leqslant p)$ be two non-disjoint stable occurrences of powers of A in W. If they overlap incorrectly then $A^2 = u \circ A \circ v$ for some elements $u, v \in G[X]$. This implies that $A = u \circ v = v \circ u$ and hence u and v are (non-trivial) powers of some element in $G[X]$. Since A is not a proper power it follows that $u = 1$ or $v = 1$ - contradiction. This shows that A^q and A^p overlap correctly. $\square$

Let $W \in G[X]$ and $\mathcal{O} = \mathcal{O}(W, A) = \{A^{q_1}, \ldots, A^{q_k}\}$ be a set of pair-wise disjoint stable occurrences of powers of a period A in W (listed according to their appearance in W from the left to the right). Then $\mathcal{O}$ induces an $\mathcal{O}$-decomposition of W of the following form:

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1} \tag{9}$$

For example, let P be a property of words (or just a property of occurrences in W) such that if two powers of A (two occurrences of powers of A in W) satisfy P and overlap correctly then their union also satisfies P. We refer to such P as *preserving correct overlappings*. In this event, by $\mathcal{O}_P = \mathcal{O}_P(W, A)$ we denote the uniquely defined set of all maximal stable occurrences of powers of A in W which satisfy the property P. Notice, that occurrences in $\mathcal{O}_P$ are pair-wise disjoint by Lemma 4.27. Thus, if P holds on every power of A then $\mathcal{O}_P(W, A) = \mathcal{O}(W, A)$ contains all maximal stable occurrences of powers of A in W. In this case, the decomposition (9) is unique and it is called the *canonical (stable)* A-decomposition of W.

The following example provides another property P that will be in use later. Let N be a positive integer and let P_N be the property of A^q that $|q| \geqslant N$. Obviously, P_N preserves correct overlappings. In this case the set $\mathcal{O}_{P_N}$ provides the so-called *canonical* N*-large* A-decompositions of W which are also uniquely defined.

Definition 4.28. Let

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}$$

be the decomposition (9) of W above. Then the numbers

$$\max_A(W) = \max\{q_i \mid i = 1, \ldots, k\}, \quad \min_A(W) = \min\{q_i \mid i = 1, \ldots, k\}$$

are called, correspondingly, the *upper* and the *lower* A-bounds of W.

Definition 4.29. Let A be a period in $G[X]$ and $W \in G[X]$. For a positive integer N we say that the N-large A-decomposition of W

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}$$

has A-size (l, r) if $\min_A(W) \geqslant l$ and $\max_A(B_i) \leqslant r$ for every $i = 1, \ldots, k$.

Let $\mathcal{A} = \{A_1, A_2, \ldots, \}$ be a sequence of periods from $G[X]$. We say that a word $W \in G[X]$ has $\mathcal{A}$-rank j ($\mathrm{rank}_{\mathcal{A}}(W) = j$) if W has a stable occurrence of $(A_j^{\pm 1})^q$ ($q \geqslant 1$) and j is maximal with this property. In this event, A_j is called the $\mathcal{A}$*-leading term* (or just the *leading term*) of W (notation $LT_{\mathcal{A}}(W) = A_j$ or $LT(W) = A_j$).

We now fix an arbitrary sequence $\mathcal{A}$ of periods in the group $G[X]$. For a period $A = A_j$ one can consider canonical A_j-decompositions of a word W and define the corresponding A_j-bounds and A_j-size. In this case we, sometimes, omit A in the writings and simply write $max_j(W)$ or $min_j(W)$ instead of $max_{A_j}(W)$, $min_{A_j}(W)$.

In the case when $\mathrm{rank}_{\mathcal{A}}(W) = j$ the canonical A_j-decomposition of W is called the *canonical* $\mathcal{A}$-decomposition of W.

Now we turn to an analog of $\mathcal{O}$-decompositions of W with respect to "periods" which are not necessarily cyclically reduced words. Let $U = D^{-1} \circ A \circ D$, where A is a period. For a set $\mathcal{O} = \mathcal{O}(W, A) = \{A^{q_1}, \ldots, A^{q_k}\}$ as above consider the $\mathcal{O}$-decomposition of a word W

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1} \tag{10}$$

Now it can be rewritten in the form:

$$W = (B_1 D)(D^{-1} \circ A^{q_1} \circ D) \cdots (D^{-1} B_k D)(D^{-1} \circ A^{q_k} \circ D)(D^{-1} B_{k+1}).$$

Let $\varepsilon_i, \delta_i = \mathrm{sgn}(q_i)$. Since every occurrence of A^{q_i} above is stable, $B_1 = \bar{B}_1 \circ A^{\varepsilon_1}$, $B_i = (A^{\delta_{i-1}} \circ \bar{B}_i \circ A^{\varepsilon_i})$, $B_{k+1} = A^{\delta_k} \circ \bar{B}_{k+1}$ for suitable words $\bar{B}_i$. This shows that the decomposition above can be written as

$$\begin{aligned} W = (\bar{B}_1 A^{\varepsilon_1} D)(D^{-1} A^{q_1} D) \cdots (D^{-1} A^{\delta_{i-1}} \bar{B}_i A^{\varepsilon_i} D) \cdots (D^{-1} A^{q_k} D)(D^{-1} A^{\delta_k} \bar{B}_{k+1}) = \\ (\bar{B}_1 D)(D^{-1} A^{\varepsilon_1} D)(D^{-1} A^{q_1} D) \cdots (D^{-1} A^{\delta_{i-1}} D)(D^{-1} \bar{B}_i D)(D^{-1} A^{\varepsilon_i} D) \cdots \\ (D^{-1} A^{q_k} D)(D^{-1} A^{\delta_k} D)(D^{-1} \bar{B}_{k+1}) \\ = (\bar{B}_1 D)(U^{\varepsilon_1})(U^{q_1}) \cdots (U^{\delta_{k-1}})(D^{-1} \bar{B}_k D)(U^{\varepsilon_k})(U^{q_k})(U^{\delta_k})(D^{-1} \bar{B}_{k+1}). \end{aligned}$$

Observe, that the cancellation between parentheses in the decomposition above does not exceed the length $d = |D|$ of D. Using notation $w = u \circ_d v$ to indicate that the cancellation between u and v does not exceed the number d, we can rewrite the decomposition above in the following form:

$$W = (\bar{B}_1 D) \circ_d U^{\varepsilon_1} \circ_d U^{q_1} \circ_d U^{\delta_1} \circ_d \cdots \circ_d U^{\varepsilon_k} \circ_d U^{q_k} \circ_d U^{\delta_k} \circ_d (D^{-1} \bar{B}_{k+1}),$$

hence

$$W = D_1 \circ_d U^{q_1} \circ_d \cdots \circ_d D_k \circ_d U^{q_k} \circ_d D_{k+1}, \tag{11}$$

where $D_1 = \bar{B}_1 D$, $D_{k+1} = D^{-1} \bar{B}_{k+1}$, $D_i = D^{-1} \bar{B}_i D$ $(2 \leqslant i \leqslant k)$, and the occurrences U^{q_i} are stable (with respect to $\circ_d$). We will refer to this decomposition of W as U-decomposition with respect to $\mathcal{O}$ (to get a rigorous definition of U-decompositions one has to replace in the definition of the $\mathcal{O}$-decomposition of W the period A by U and $\circ$ by $\circ_{|D|}$). In the case when an A-decomposition of W (with respect to $\mathcal{O}$) is unique then the corresponding U-decomposition of W is also unique, and in this event one can easily rewrite A-decompositions of W into U-decomposition and vice versa.

We summarize the discussion above in the following lemma.

Lemma 4.30. *Let $A \in G[X]$ be a period and $U = D^{-1} \circ A \circ D \in G[X]$. Then for a word $W \in G[X]$ if*

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}$$

is a stable A-decomposition of W then

$$W = D_1 \circ_d U^{q_1} \circ_d \cdots \circ_d D_k \circ_d U^{q_k} \circ_d D_{k+1}$$

is a stable U-decomposition of W, where D_i are defined as in (11). And vice versa.

From now on we fix the following set of leading terms

$$\mathcal{A}_{L,p} = \{A_j \mid j \leqslant L, \phi = \phi_{L,p}\}$$

for a given multiple L of $K = K(m,n)$ and a given tuple p.

Definition 4.31. Let $W \in G[X]$ and N be a positive integer. A word of the type A_s is termed the N-large leading term $LT_N(W)$ of the word W if A_s^q has a stable occurrence in W for some $q \geq N$, and s is maximal with this property. The number s is called the N-rank of W $(s = \mathrm{rank}_N(W), s \geq 1)$.

Lemma 4.32. *Let $W \in G[X]$, $N \geq 2$, and let $A = LT_N(W)$. Then W can be presented in the form*

$$W = B_1 \circ A^{q_1} \circ \ldots B_k \circ A^{q_k} \circ B_{k+1} \tag{12}$$

where A^{q_i} are maximal stable occurrences, $q_i \geq N$, and $\mathrm{rank}_N(B_i) < \mathrm{rank}_N(W)$. This presentation is unique and it is called the N-large A-presentation of W.

PROOF. Existence follows from the definition of the leading term $LT_N(W)$. To prove uniqueness it is suffice to notice that two stable occurrences A^q and A^r do not intersect. Since $A = LT_N(W)$ is cyclically reduced and it is not a proper power it follows that an equality $A^2 = u \circ A \circ v$ holds in $F(X \cup C_S)$ if and only if $u = 1$ or $v = 1$. So, stable occurrences of A^q and A^r are protected from overlapping by the neighbors of A on each side of them. $\square$

In Lemmas 4.6, 4.7, 4.8, and 4.9 we described precisely the leading terms $A_j, j = 1, \ldots, K$ as the cyclically reduced forms of some words A_j. It is not easy to describe A_j for an arbitrary $j > K$. So we are not going to do it here, instead, we chose a compromise by introducing a modified version of A_j which is not cyclically reduced, in general, but which is "more cyclically reduced" then the initial word A_j.

Let L be a multiple of K and $1 \leq j \leq K$. Define

$$A^*_{L+j} = A^*(\phi_{L+j}) = A_j^{\phi_L}.$$

LEMMA 4.33. *Let L be a multiple of K and $1 \leq j \leq K$. Let $p = (p_1, \ldots, p_n)$ be $N+3$-large tuple. Then $A_{L+j} = cycred(A^*(\phi_{L+j}))$. Moreover, if*

$$A^*(\phi_{L+j}) = R^{-1} \circ A_{L+j} \circ R$$

then $rank_N(R) \leq L - K + j + 2$ and $|R| < |A_{L+j}|$.

PROOF. First, let $L = K$. Consider elementary periods $x_i = A_{m+4i-3}$ and $A_1 = c_1^{z_1} c_2^{z_2}$. For $i \neq n$, $x_i^{2\phi_K} = x_i^{\phi_k} \circ x_i^{\phi_K}$. For $i = n$,

$$A^*(\phi_{K+m+4n-3}) = R^{-1} \circ A_{K+m+4n-3} \circ R,$$

where $R = A_{m+4i-4}^{p_{m+4n-4}}$, therefore $\mathrm{rank}_N(R) = m + 4n - 4$. For the other elementary period, $(c_1^{z_1} c_2^{z_2})^{2\phi_K} = (c_1^{z_1} c_2^{z_2})^{\phi_K} \circ (c_1^{z_1} c_2^{z_2})^{\phi_K}$.

Any other A_j can be written in the form $A_j = u_1 \circ v_1 \circ u_2 \circ v_2 \circ u_3$, where v_1, v_2 are the first and the last elementary squares in A_j, which are parts of big powers of elementary periods. The Nielsen property of ϕ_K implies that the word R for $A^*(\phi_{K+j})$ is the word that cancels between $(v_2 u_3)^{\phi_K}$ and $(u_1 v_1)^{\phi_K}$. It definitely has N-large rank $\leq K$, because the element $(v_2 u_3 u_1 v_1)^{\phi_K}$ has N-large rank $\leq K$. To give an exact bound for the rank of R we consider all possibilities for A_j:

(1) A_i begins with z_i^{-1} and ends with z_{i+1}, $i = 1, \ldots, m-1$,
(2) A_m begins with z_m^{-1} and ends with x_1^{-1},
(3) A_{m+4i-4} begins with $x_{i-1} y_{i-2}^{-1} x_{i-2}^{-2}$, if $i = 3, \ldots n$, and ends with $x_{i-1}^2 y_{i-1} x_i^{-1}$ if $i = 2, \ldots, n$, If $i = 2$ it begins with $x_1 \Pi_{j=m}^{1} c_j^{-z_j} (c_2^{-z_2} c_1^{-z_1})^2$.
(4) A_{m+4i-2} and A_{m+4i-1} begins with $x_i y_{i-1}^{-1} x_{i-1}^{-2}$ and ends with $x_i^2 y_i$ if $i = 1, \ldots, n$.

Therefore, $A_i^{\phi_K}$ begins with z_{i+1}^{-1} and ends with z_{i+2}, $i = 1, \ldots, m-2$, and is cyclically reduced.

$A_{m-1}^{\phi_K}$ begins with z_m^{-1} and ends with x_1, and is cyclically reduced, $A_m^{\phi_K}$ begins with z_m^{-1} and ends with x_1^{-1} and is cyclically reduced.

We have already considered $A_{m+4i-3}^{\phi_K}$.

Elements $A_{m+4i-4}^{\phi_K}, A_{m+4i-2}^{\phi_K}, A_{m+4i-1}^{\phi_K}$ are not cyclically reduced. By Lemma 4.21, for $A^*(\phi_K + m + 4i - 4)$, one has $R = (x_{i-1} y_{i-2}^{-1})^{\phi_K}$ $(rank_N(R) = m + 4i - 4)$; for $A^*(\phi_K + m + 4i - 2)$, and $A^*(\phi_K + m + 4i - 2)$, $R = (x_i y_{i-1}^{-1})^{\phi_K}$ $(rank_N(R) = m + 4i)$.

This proves the statement of the Lemma for $L = K$.

We can suppose by induction that $A^*(\phi_{L-K+j}) = R^{-1} \circ A_{L-K+j} \circ R$, and $rank_N(R) \leq L - 2K + j + 2$. The cancellations between $A_{L-K+j}^{\phi_K}$ and R^{ϕ_K} and between $A_{L-K+j}^{\phi_K}$ and $A_{L-K+j}^{\phi_K}$ correspond to cancellations in words u^{ϕ_K}, where u is a word in $\mathcal{W}_\Gamma$ between two elementary squares. These cancellations are in rank $\leq K$, and the statement of the lemma follows. □

LEMMA 4.34. *Let $W \in F(X \cup C_S)$ and $A = A_j = LT_N(W)$, and $A^* = R^{-1} \circ A \circ R$. Then W can be presented in the form*

$$W = B_1 \circ_d A^{*q_1} \circ_d B_2 \circ_d \cdots \circ_d B_k \circ_d A^{*q_k} \circ_d B_{k+1} \tag{13}$$

where A^{*q_i} are maximal stable N-large occurrences of A^* in W and $d \leq |R|$. This presentation is unique and it is called the canonical N-large A^*-decomposition of W.

PROOF. The result follows from existence and uniqueness of the canonical A-decompositions. Indeed, if

$$W = B_1 \circ A^{q_1} \circ B_2 \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1}$$

is the canonical A-decomposition of W, then

$$(B_1R)(R^{-1}AR)^{q_1}(R^{-1}B_2R)\cdots(R^{-1}B_kR)(R^{-1}AR)^{q_k}(R^{-1}B_{k+1})$$

is the canonical A^*-decomposition of W. Indeed, since every A^{q_i} is a stable occurrence, then every B_i starts with A (if $i \neq 1$) and ends with A (if $i = k+1$). Hence $R^{-1}B_iR = R^{-1} \circ B_i \circ R$.

Conversely if

$$W = B_1A^{*q_1}B_2\cdots B_kA^{*q_k}B_{k+1}$$

is an A^*-representation of W then

$$W = (B_1R^{-1}) \circ A^{q_1} \circ (RB_2R^{-1}) \circ \cdots \circ (RB_kR^{-1}) \circ A^{q_k} \circ (RB_{k+1})$$

is the canonical A-decomposition for W. □

In the following lemma we collect various properties of words $x_i^{\phi_L}, y_i^{\phi_L}, z_j^{\phi_L}$ where $L = Kl$ is a multiple of K.

LEMMA 4.35. *Let $X = \{x_i, y_i, z_j \mid i = 1, \ldots, n, j = 1, \ldots, m\}$, let $K = K(m,n)$, and $L = Kl$ be a multiple of K. Then for any number $N \geq 5$ and for any N-large tuple $p \in N^L$ the following holds (below $\phi = \phi_{L,p}$, $A_j = A_j$):*

1. *If $i < j \leq L$ then A_j^2 does not occur in A_i;*
2. *Let $i \leq K$, $j = i + L$. There are positive integers s, $1 \leq j_1, \ldots, j_s \leq j$, integers $\varepsilon_1, \ldots, \varepsilon_s$ with $|\varepsilon_t| \leq 3$, and words $w_1, \ldots, w_{s+1} \in F(X \cup C_S)$ (which do not depend on the tuple p and do not contain any square of leading terms) such that the leading term A_j (A_j^*) of ϕ_j has the following form:*

$$w_1 \circ A_{j_1}^{p_{j_1}+\varepsilon_1} \circ w_2 \circ \cdots \circ w_s \circ A_{j_s}^{p_{j_s}+\varepsilon_s} \circ w_{s+1}, \tag{14}$$

   i.e., the "periodic structure" of A_j (A_j^*) does not depend on the tuple p.
3. *Let $i \leq K$, $u \in \mathcal{W}_{\Gamma,L}$ such that*

$$u = v_1 \circ A_{j_1}^{p_{j_1}+\varepsilon_1} \circ v_2 \circ \cdots \circ v_r \circ A_{j_r}^{p_{j_r}+\varepsilon_r} \circ v_{r+1},$$

where $j_1, \dots, j_r \leq i$, and at least one of j_t is equal to i, $|\varepsilon_t| \leq 1$, and words $v_1, \dots, v_{r+1} \in F(X \cup C_S)$ do not depend on p. Then

$$u^{\phi_L} = v_1^{\phi_L} A_{j_1}^{\sigma_1 \phi_L} W_1^{-1} \circ A_{j_1+L}^{(p_{j_1}+\varepsilon_1-2\sigma_1)} \circ W_1 A_{j_1}^{\sigma_1 \phi_L} v_2^{\phi_L} \ \dots$$

$$v_r^{\phi_L} A_{j_r}^{\sigma_r \phi_L} W_r^{-1} \circ A_{j_r+L}^{(p_{j_r}+\varepsilon_r-2\sigma_r)} \circ W_r A_{j_r}^{\sigma_r \phi_L} v_{r+1}^{\phi L},$$

where $A_{j_t}^{\phi_L} = W_t^{-1} \circ A_{j_t+L} \circ W_t$; $\sigma_t = 1$ if p_t is positive and $\sigma_t = -1$ if p_t is negative. In addition, for each $t = 1, \dots, r$ the product

$$W_t A_{j_t}^{\sigma_t \phi_L} v_{t+1}^{\phi_L} A_{j_{t+1}}^{\sigma_{t+1} \phi_L} W_{t+1}^{-1}$$

has form (14) with $j_1, \dots, j_s < i + L$.

(4) *For any $i \leq K$ and any $x \in X^{\pm 1}$ there is a positive integer s and there are indices $1 \leq j_1, \dots, j_s \leq i$, integers $\varepsilon_1, \dots, \varepsilon_s$ with $|\varepsilon_t| \leq 1$, and words $w_1, \dots, w_{s+1} \in F(X \cup C_S)$ which do not depend on the tuple p such that the element x^{ϕ_i} can be presented in the following form:*

$$x^{\phi_i} = w_1 \circ A_{j_1}^{p_{j_1}+\varepsilon_1} \circ w_2 \circ \dots \circ w_s \circ A_{j_s}^{p_{j_s}+\varepsilon_s} \circ w_{s+1}.$$

Proof. Statement (1) follows from Lemmas 4.6–4.8.

Statements (2) and (3) will be proved by simultaneous induction on $j = i + L$. Case $l = 0$ corresponds to $i \leq K$. In this case statement (2) follows from Lemmas 4.6 - 4.8 and statement (3) is simply the assumption of the lemma. A_i has form (14) with $j_1, \dots, j_s < i$ and $|\varepsilon_1|, \dots, |\varepsilon_s| \leq 1$.

We know that A_{j_t} contains an elementary square (actually, big power) for any $t = 1, \dots s$, $A_{j_t}^{\phi_K} = R_{j_t}^{-1} \circ A_{j_t+K} \circ R_{j_t}$, where R_{j_t} does not contain big powers of A_k for $k \geq j_t + 2$. Then it follows from the second statement of Lemma 4.21 that

$$A_i^{\phi_K} = w_1^{\phi_K} R_{j_1}^{-\sigma_1} A_{j_1+K}^{\sigma_1} \circ A_{j_1+K}^{p_{j_1}+\varepsilon_1-2\sigma_1} \circ A_{j_1+K}^{\sigma_1} R_{j_1}^{\sigma_1} w_2^{\phi_K} \ \dots \ w_s^{\phi_K} R_{j_s}^{-\sigma_s} A_{j_s+K}^{\sigma_s} \circ A_{j_s+K}^{p_{j_s}+\varepsilon_s-2\sigma_s} \circ A_{j_s+K}^{\sigma_s} R_{j_s}^{\sigma_s} w_{s+1}^{\phi_K},$$

where $\sigma_t = 1$ if p_{j_t} is positive and $\sigma_t = -1$ if p_{j_t} is negative.

When we apply ϕ_K, the images of elementary big powers in A_{j_t} by Lemma 4.21 are not touched by cancellations between $w_{t-1}^{\phi_K}$ and $A_{j_t}^{\phi_K}$, and between $A_{j_t}^{\phi_K}$ and $w_{t+1}^{\phi_K}$, therefore $A_i^{\phi_L} =$

$$w_1^{\phi_L} R_{j_1}^{-\sigma_1 \phi_{L-K}} A_{j_1+K}^{\sigma_1 \phi_{L-K}} W_1^{-1} \circ A_{j_1+L}^{p_{j_1}+\varepsilon_1-2\sigma_1} \circ W_1 A_{j_1+K}^{\sigma_1 \phi_{L-K}} R_{j_1}^{\sigma_1 \phi_{L-K}} w_2^{\phi_L} \ \dots$$

$$w_s^{\phi_L} R_{j_s}^{-\sigma_s \phi_{L-K}} A_{j_s+K}^{\sigma_s \phi_{L-K}} W_s^{-1} \circ A_{j_s+L}^{p_{j_s}+\varepsilon_s-2\sigma_s} \circ W_s A_{j_s+K}^{\sigma_s \phi_{L-K}} R_{j_s}^{\sigma_s \phi_{L-K}} w_{s+1}^{\phi_L},$$

where $A_{j_t+K}^{\phi_{L-K}} = W_t^{-1} \circ A_{j_t+L} \circ W_t$, $\sigma_t = 1$ if p_{j_t} is positive and $\sigma_t = -1$ if p_{j_t} is negative ($t = 1, \dots, s,$). We can now apply statement 3) for $i_1 + Kl, i_1 < i$ to elements
$w_1^{\phi_L} R_{j_1}^{-\sigma_1 \phi_{L-K}} A_{j_1+K}^{\sigma_1 \phi_{L-K} W_1^{-1}}, \dots, \ W_s A_{j_s+K}^{\sigma_s \phi_{L-K}} R_{j_s}^{\sigma_s \phi_{L-K}} w_{s+1}^{\phi_L}$.

To prove statement (3) for $i + Kl$, we use it for $i_1 + Kl$ and statement (2) for $i + Kl$.

(4) Existence of such a decomposition follows from Lemmas 4.6–4.8. □

Corollary 4.36. *If L is a multiple of K, then the automorphism ϕ_L satisfies the Nielsen property with respect to $\bar{\mathcal{W}}_\Gamma$ with exceptions $E(n, m)$.*

Proof. The middles M_x of elements from X and from $E(m,n)$ with respect to ϕ_K contain big powers of some A_j, where $j = 1, \ldots, K$. By Lemma 4.35 these big powers cannot disappear after application of ϕ_{L-K}. Therefore, $M_x^{\phi_{L-K}}$ contains the middle of x with respect to ϕ_L. $\square$

Corollary 4.37. *Let $u, v \in \bar{\mathcal{W}}_\Gamma$. If the canceled subword in the product $u^{\phi_K} v^{\phi_K}$ does not contain A_j^l for some $j \leq K$ and $l \in \mathbb{Z}$ then the canceled subword in the product $u^{\phi_{K+L}} v^{\phi_{K+L}}$ does not contain the subword A_{L+j}^l.*

Lemma 4.38. *Let $W \in \mathcal{W}_{\Gamma,L}$. Suppose that $1 \leq r \leq K$, L_1 is a multiple of K, and $j = r + L_1$. Then the following conditions are equivalent:*

1) *$rank_N(W) = r$ and*
$$W = D_1 \circ A_r^{q_1} \circ D_2 \ldots D_k \circ A_r^{q_k} \circ D_{k+1}$$
is a stable 5-large A_r-decomposition of W;

2) *$rank_N(W^{\phi_{L_1}}) = j$ and*
$$W^{\phi_{L_1}} = (D_1 A_j^{\varepsilon_1})^{\phi_{L_1}} \circ_d A_j^{*q_1 - \varepsilon_1 - \delta_1} \circ_d (A_j^{\delta_1} D_2 A_j^{\varepsilon_2})^{\phi_{L_1}} \ldots$$
$$(A_j^{\delta_{k-1}} D_k A_j^{\varepsilon_k})^{\phi_{L_1}} \circ_d A_j^{*q_k - \varepsilon_k - \delta_k} \circ (A_j^{\delta_k} D_{k+1})^{\phi_{L_1}}$$
is a stable A_j^*-decomposition of $W^{\phi_{L_1}}$, where $\delta_s, \varepsilon_s \in \{0, \pm 1\}$ depending on the sign of q_s and β.

Proof. It follows from Lemmas 4.34 and 4.35. Indeed, let $W \in \mathcal{W}_{\Gamma,L}$ and
$$W = D_1 \circ A_r^{q_1} \circ D_2 \ldots D_k \circ A_r^{q_k} \circ D_{k+1}$$
the canonical N-large A_r-decomposition of W. Then by Lemma 4.35 (3)
$$W^{\phi_{L_1}} = (D_1 A_r^{\sigma_1})^{\phi_{L_1}} w_r^{-1} \circ A_j^{q_1 - 2\sigma_1} \circ w_r (A_r^{\sigma_1} D_2 A_r^{\sigma_2})^{\phi_{L_1}} \ldots$$
$$(A_r^{\sigma_{k-1}} D_k A_r^{\sigma_k})^{\phi_{L_1}} w_r^{-1} \circ A_j^{q_k - 2\sigma_k} \circ w_r (A_r^{\sigma_k} D_{k+1})^{\phi_{L_1}}$$
where $A_r^{\phi_{L_1}} = w_r^{-1} \circ A_j \circ w_r$, $\sigma_t \in \{1, -1\}$. This implies that the canonical A^*-decomposition of $W^{\phi_{L_1}}$ takes the form described in 2).

Conversely, suppose 2) is the canonical A^*-decomposition of $W^{\phi_{L_1}}$, but 1) is not the canonical A_r-decomposition of W. Then taking the canonical A_r-decomposition of W and applying ϕ_{L_1} by 1) we get another canonical decomposition of $W^{\phi_{L_1}}$ - contradiction with uniqueness of A^*-decompositions. $\square$

Lemma 4.39. *Suppose p is an $(N+3)$-large tuple, $\phi_j = \phi_{jp}$. Let L be a multiple of K. Then:*

(1) (a) *$x_i^{\phi_j}$ has a canonical N-large A_j^*-decomposition of size $(N,2)$ if either $j \equiv m + 4(i-1) (mod\ K)$, or $j \equiv m + 4i - 2 (mod\ K)$, or $j \equiv m + 4i (mod\ K)$. In all other cases $rank(x_i^{\phi_j}) < j$.*

(b) *$y_i^{\phi_j}$ has a canonical N-large A_j^*-decomposition of size $(N,2)$ if either $j \equiv m + 4(i-1) (mod\ K)$, or $j \equiv m + 4i - 3 (mod\ K)$, or $j \equiv m + 4i - 1 (mod\ K)$, or $j \equiv m + 4i\ (mod\ K)$. In all other cases $rank(y_i^{\phi_j}) < j$.*

(c) *$z_i^{\phi_j}$ has a canonical N-large A_j^*-decomposition of size $(N,2)$ if $j \equiv i\ (mod\ K)$ and either $1 \leq i \leq m-1$ or $i = m$ and $n \neq 0$. In all other cases $rank(z_i^{\phi_j}) < j$.*

(d) *if* $n = 0$ *then* $z_m^{\phi_j}$ *has a canonical* N*-large* A_j^**-decomposition of size* $(N, 2)$ *if* $j \equiv m - 1$ $(mod\ K)$*. In all other cases* $rank(z_m^{\phi_j}) < j$*.*

(2) *If* $j = r + L$, $0 < r \leq K$, $(w_1 \dots w_k) \in Sub_k(X^{\pm\gamma_K \cdots \gamma_{r+1}})$ *then either* $(w_1 \dots w_k)^{\phi_j} = (w_1 \dots w_k)^{\phi_{j-1}}$*, or* $(w_1 \dots w_k)^{\phi_j}$ *has a canonical* N*-large* A_j^**-decomposition. In any case,* $(w_1 \dots w_k)^{\phi_j}$ *has a canonical* N*-large* A_s^**-decomposition in some rank* s, $j - K + 1 \leq s \leq j$*.*

Proof. (1) Consider $y_i^{\phi_{L+m+4i}}$:

$$y_i^{\phi_{L+m+4i}} = (x_{i+1}^{\phi_L} y_i^{-\phi_{L+m+4i-1}})^{q_4-1} x_{i+1}^{\phi_L} (y_i^{\phi_{L+m+4i-1}} x_{i+1}^{-\phi_L})^{q_4},$$

In this case $A^*(\phi_{L+m+4i}) = x_{i+1}^{\phi_{L+m+4i-1}} y_i^{-\phi_{L+m+4i-1}}$.

To write a formula for $x_i^{\phi_{L+m+4i}}$, denote $\tilde{y}_{i-1} = y_{i-1}^{\phi_{L+m+4i-5}}$, $\bar{x}_i = x_i^{\phi_L}$, $\bar{y}_i = y_i^{\phi_L}$. Then

$$\begin{aligned} x_i^{\phi_{L+m+4i}} = {} & (\bar{x}_{i+1} y_i^{-\phi_{L+m+4i-1}})^{q_4-1} \bar{x}_{i+1} \\ & (((\bar{x}_i \tilde{y}_{i-1}^{-1})^{q_0} \bar{x}_i^{q_1} \bar{y}_i)^{q_2-1} (\bar{x}_i \tilde{y}_{i-1}^{-1})^{q_0} \bar{x}_i^{q_1+1} \bar{y}_i)^{-q_3+1} \bar{y}_i^{-1} \bar{x}_i^{-q_1} (\tilde{y}_{i-1} \bar{x}_i^{-1})^{q_0}. \end{aligned}$$

Similarly we consider $z_i^{\phi_{L+i}}$.

(2) If in a word $(w_1 \cdots w_k)^{\phi_j}$ all the powers of $A_j^{p_j}$ are cancelled (by Lemma 4.35 they can only cancel completely and the process of cancellations does not depend on p) then if we consider an A_j^*-decomposition of $(w_1 \cdots w_k)^{\phi_j}$, all the powers of A_j^* are also completely cancelled. By construction of the automorphisms γ_j, this implies that $(w_1 \cdots w_k)^{\gamma_j \phi_{j-1}} = (w_1 \cdots w_k)^{\phi_{j-1}}$. □

5. Generic solutions of orientable quadratic equations

Let G be a finitely generated fully residually free group and $S = 1$ a standard quadratic orientable equation over G which has a solution in G. In this section we effectively construct discriminating sets of solutions of $S = 1$ in G. The main tool in this construction is an embedding

$$\lambda : G_{R(S)} \to G(U, T)$$

of the coordinate group $G_{R(S)}$ into a group $G(U, T)$ which is obtained from G by finitely many extensions of centralizers. There is a nice set Ξ_P (see Section 1.4 in [**16**]) of discriminating G-homomorphisms from $G(U, T)$ onto G. The restrictions of homomorphisms from Ξ_P onto the image $G_{R(S)}^\lambda$ of $G_{R(S)}$ in $G(U, T)$ give a discriminating set of G-homomorphisms from $G_{R(S)}^\lambda$ into G, i.e., solutions of $S = 1$ in G. This idea was introduced in [**12**] to describe the radicals of quadratic equations.

It has been shown in [**12**] that the coordinate groups of non-regular standard quadratic equations $S = 1$ over G are already extensions of centralizers of G, so in this case we can immediately put $G(U, T) = G_{R(S)}$ and the result follows. Hence we can assume from the beginning that $S = 1$ is regular.

Notice, that all regular quadratic equations have solutions in general position, except for the equation $[x_1, y_1][x_2, y_2] = 1$ (see [**13**], Section 2).

For the equation $[x_1, y_1][x_2, y_2] = 1$ we do the following trick. In this case we view the coordinate group $G_{R(S)}$ as the coordinate group of the equation $[x_1, y_1] =$

$[y_2, x_2]$ over the group of constants $G * F(x_2, y_2)$. So the commutator $[y_2, x_2] = d$ is a non-trivial constant and the new equation is of the form $[x, y] = d$, where all solutions are in general position. Therefore, we can assume that $S = 1$ is one of the following types (below d, c_i are nontrivial elements from G):

$$\prod_{i=1}^{n}[x_i, y_i] = 1, \quad n \geqslant 3; \tag{15}$$

$$\prod_{i=1}^{n}[x_i, y_i] \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n \geqslant 1, m \geqslant 0; \tag{16}$$

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad m \geqslant 2, \tag{17}$$

and it has a solution in G in general position.

Observe, that since $S = 1$ is regular then Nullstellenzats holds for $S = 1$, so $R(S) = ncl(S)$ and $G_{R(S)} = G[X]/ncl(S) = G_S$.

For a group H and an element $u \in H$ by $H(u, t)$ we denote the extension of the centralizer $C_H(u)$ of u:

$$H(u, t) = \langle H, t \mid t^{-1} x t = x \ \ (x \in C_H(u)) \rangle.$$

If

$$G = G_1 \leqslant G_1(u_1, t_1) = G_2 \leqslant \ldots \leqslant G_n(u_n, t_n) = G_{n+1}$$

is a chain of extensions of centralizers of elements $u_i \in G_i$, then we denote the resulting group G_{n+1} by $G(U, T)$, where $U = \{u_1, \ldots, u_n\}$ and $T = \{t_1, \ldots, t_n\}$.

Let $\beta : G_{R(S)} \to G$ be a solution of the equation $S(X) = 1$ in the group G such that

$$x_i^\beta = a_i, y_i^\beta = b_i, z_i^\beta = e_i.$$

Then

$$d = \prod_{i=1}^{m} e_i^{-1} c_i e_i \prod_{i=1}^{n}[a_i, b_i].$$

Hence we can rewrite the equation $S = 1$ in the following form (for appropriate m and n):

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n}[x_i, y_i] = \prod_{i=1}^{m} e_i^{-1} c_i e_i \prod_{i=1}^{n}[a_i, b_i]. \tag{18}$$

Proposition 5.1. *Let $S = 1$ be a regular quadratic equation* (18) *and $\beta : G_{R(S)} \to G$ a solution of $S = 1$ in G in a general position. Then one can effectively construct a sequence of extensions of centralizers*

$$G = G_1 \leqslant G_1(u_1, t_1) = G_2 \leqslant \ldots \leqslant G_n(u_n, t_n) = G(U, T)$$

and a G-homomorphism $\lambda_\beta : G_{R(S)} \to G(U, T)$.

Proof. By induction we define a sequence of extensions of centralizers and a sequence of group homomorphisms in the following way.

Case: $m \neq 0, n = 0$. In this event for each $i = 1, \ldots, m-1$ we define by induction a pair (θ_i, H_i), consisting of a group H_i and a G-homomorphism $\theta_i : G[X] \to H_i$.

Before we will go into formalities let us explain the idea that lies behind this. If $z_1 \to e_1, \ldots, z_m \to e_m$ is a solution of an equation

$$z_1^{-1} c_1 z_1 \ldots z_m^{-1} c_m z_m = d, \tag{19}$$

then transformations

$$e_i \to e_i \left(c_i^{e_i} c_{i+1}^{e_{i+1}}\right)^q, \quad e_{i+1} \to e_{i+1} \left(c_i^{e_i} c_{i+1}^{e_{i+1}}\right)^q, \quad e_j \to e_j \quad (j \neq i, i+1), \tag{20}$$

produce a new solution of the equation (19) for an arbitrary integer q. This solution is composition of the automorphism γ_i^q and the solution e. To avoid collapses under cancellation of the periods $(c_i^{e_i} c_{i+1}^{e_{i+1}})^q$ (which is an important part of the construction of the discriminating set of homomorphisms Ξ_P in Section 1.4 in [**16**]) one might want to have number q as big as possible, the best way would be to have $q = \infty$. Since there are no infinite powers in G, to realize this idea one should go outside the group G into a bigger group, for example, into an ultrapower G' of G, in which a non-standard power, say t, of the element $c_i^{e_i} c_{i+1}^{e_{i+1}}$ exists. It is not hard to see that the subgroup $\langle G, t\rangle \leqslant G'$ is an extension of the centralizer $C_G(c_i^{e_i} c_{i+1}^{e_{i+1}})$ of the element $c_i^{e_i} c_{i+1}^{e_{i+1}}$ in G. Moreover, in the group $\langle G, t\rangle$ the transformation (20) can be described as

$$e_i \to e_i t, \quad e_{i+1} \to e_{i+1} t, \quad e_j \to e_j \quad (j \neq i, i+1), \tag{21}$$

Now, we are going to construct formally the subgroup $\langle G, t\rangle$ and the corresponding homomorphism using (21).

Let H be an arbitrary group and $\beta : G_S \to H$ a homomorphism. Composition of the canonical projection $G[X] \to G_S$ and β gives a homomorphism $\beta_0 : G[X] \to H$. For $i = 0$ put

$$H_0 = H, \quad \theta_0 = \beta_0$$

Suppose now, that a group H_i and a homomorphism $\theta_i : G[X] \to H_i$ are already defined. In this event we define H_{i+1} and θ_{i+1} as follows

$$H_{i+1} = \left\langle H_i, r_{i+1} \mid \left[C_{H_i}(c_{i+1}^{z_{i+1}^{\theta_i}} c_{i+2}^{z_{i+2}^{\theta_i}}), r_{i+1}\right] = 1 \right\rangle,$$

$$z_{i+1}^{\theta_{i+1}} = z_{i+1}^{\theta_i} r_{i+1}, \quad z_{i+2}^{\theta_{i+1}} = z_{i+2}^{\theta_i} r_{i+1}, \quad z_j^{\theta_{i+1}} = z_j^{\theta_i}, \quad (j \neq i+1, i+2).$$

By induction we constructed a series of extensions of centralizers

$$G = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_{m-1} = H_{m-1}(G)$$

and a homomorphism

$$\theta_{m-1,\beta} = \theta_{m-1} : G[X] \to H_{m-1}(G).$$

Observe, that,

$$c_{i+1}^{z_{i+1}^{\theta_i}} c_{i+2}^{z_{i+2}^{\theta_i}} = c_{i+1}^{e_{i+1} r_i} c_{i+2}^{e_i}$$

so the element r_{i+1} extends the centralizer of the element $c_{i+1}^{e_{i+1} r_i} c_{i+2}^{e_i}$. In particular, the following equality holds in the group $H_{m-1}(G)$ for each $i = 0, \ldots, m-1$:

$$[r_{i+1}, c_{i+1}^{e_{i+1} r_i} c_{i+2}^{e_i}] = 1. \tag{22}$$

(where $r_0 = 1$). Observe also, that

$$z_1^{\theta_{m-1}} = e_1 r_1, \quad z_i^{\theta_{m-1}} = e_i r_{i-1} r_i, \quad z_m^{\theta_{m-1}} = e_m r_{m-1} \quad (0 < i < m). \tag{23}$$

From (22) and (23) it readily follows that

$$\left(\prod_{i=1}^{m} z_i^{-1} c_i z_i\right)^{\theta_{m-1}} = \prod_{i=1}^{m} e_i^{-1} c_i e_i, \tag{24}$$

so θ_{m-1} gives rise to a homomorphism (which we again denote by θ_{m-1} or θ_β)

$$\theta_{m-1} : G_S \longrightarrow H_{m-1}(G).$$

Now we iterate the construction one more time replacing H by $H_{m-1}(G)$ and β by θ_{m-1} and put:

$$H_\beta(G) = H_{m-1}(H_{m-1}(G)), \quad \lambda_\beta = \theta_{\theta_{m-1}} : G_S \longrightarrow H_\beta(G).$$

The group $H_\beta(G)$ is union of a chain of extensions of centralizers which starts at the group H.

If $H = G$ then all the homomorphisms above are G-homomorphisms. Now we can write

$$H_\beta(G) = G(U, T)$$

where $U = \{u_1, \ldots, u_{m-1}, \bar{u}_1, \ldots, \bar{u}_{m-1}\}$, $T = \{r_1, \ldots, r_{m-1}, \bar{r}_1, \ldots, \bar{r}_{m-1}\}$ and $\bar{u}_i, \bar{r}_i$ are the corresponding elements when we iterate the construction:

$$u_{i+1} = c_{i+1}^{e_{i+1} r_i} c_{i+2}^{e_{i+2}}, \quad \bar{u}_{i+1} = c_{i+1}^{e_{i+1} r_i r_{i+1} \bar{r}_i} c_{i+2}^{e_{i+2} r_{i+1} r_{i+2}}.$$

Case: $m = 0, n > 0$. In this case $S = [x_1, y_1] \cdots [x_n, y_n] d^{-1}$. Similar to the case above we start with the principal automorphisms. They consist of two Dehn's twists:

$$x \to y^p x, \quad y \to y; \tag{25}$$

$$x \to x, \quad y \to x^p y; \tag{26}$$

which fix the commutator $[x, y]$, and the third transformation which ties two consequent commutators $[x_i, y_i][x_{i+1}, y_{i+1}]$:

$$x_i \to (y_i x_{i+1}^{-1})^{-q} x_i, \quad y_i \to (y_i x_{i+1}^{-1})^{-q} y_i (y_i x_{i+1}^{-1})^q, \tag{27}$$

$$x_{i+1} \to (y_i x_{i+1}^{-1})^{-q} x_{i+1} (y_i x_{i+1}^{-1})^q, \quad y_{i+1} \to (y_i x_{i+1}^{-1})^{-q} y_{i+1}.$$

Now we define by induction on i, for $i = 0, \ldots, 4n-1$, pairs (G_i, α_i) of groups G_i and G-homomorphisms $\alpha_i : G[X] \to G_i$. Put

$$G_0 = G, \quad \alpha_0 = \beta.$$

For each commutator $[x_i, y_i]$ in $S = 1$ we perform consequently three Dehn's twists (26), (25), (26) (more precisely, their analogs for an extension of a centralizer) and an analog of the connecting transformation (27) provided the next commutator exists. Namely, suppose G_{4i} and α_{4i} have been already defined. Then

$$\begin{aligned}
G_{4i+1} &= \left\langle G_{4i}, t_{4i+1} \mid [C_{G_{4i}}(x_{i+1}^{\alpha_{4i}}), t_{4i+1}] = 1 \right\rangle; \\
y_{i+1}^{\alpha_{4i+1}} &= t_{4i+1} y_{i+1}^{\alpha_{4i}}, \quad s^{\alpha_{4i+1}} = s^{\alpha_{4i}} \quad (s \neq y_{i+1}). \\
G_{4i+2} &= \left\langle G_{4i+1}, t_{4i+2} \mid [C_{G_{4i+1}}(y_{i+1}^{\alpha_{4i+1}}), t_{4i+2}] = 1 \right\rangle; \\
x_{i+1}^{\alpha_{4i+2}} &= t_{4i+2} x_{i+1}^{\alpha_{4i+1}}, \quad s^{\alpha_{4i+2}} = s^{\alpha_{4i+1}} \quad (s \neq x_{i+1}); \\
G_{4i+3} &= \left\langle G_{4i+2}, t_{4i+3} \mid \left[C_{G_{4i+2}}(x_{i+1}^{\alpha_{4i+2}}), t_{4i+3}\right] = 1 \right\rangle; \\
y_{i+1}^{\alpha_{4i+3}} &= t_{4i+3} y_{i+1}^{\alpha_{4i+2}}, \quad s^{\alpha_{4i+3}} = s^{\alpha_{4i+2}} \quad (s \neq y_{i+1}); \\
G_{4i+4} &= \left\langle G_{4i+3}, t_{4i+4} \mid \left[C_{G_{4i+3}}\left(y_{i+1}^{\alpha_{4i+3}} x_{i+2}^{-\alpha_{4i+3}}\right), t_{4i+4}\right] = 1 \right\rangle; \\
x_{i+1}^{\alpha_{4i+4}} &= t_{4i+4}^{-1} x_{i+1}^{\alpha_{4i+3}}, y_{i+1}^{\alpha_{4i+4}} = y_{i+1}^{\alpha_{4i+3} t_{4i+4}}, x_{i+2}^{\alpha_{4i+4}} = x_{i+2}^{\alpha_{4i+3} t_{4i+4}}, \\
y_{i+2}^{\alpha_{4i+4}} &= t_{4i+4}^{-1} y_{i+2}^{\alpha_{4i+3}}; \\
s^{\alpha_{4i+4}} &= s^{\alpha_{4i+3}} \quad (s \neq x_{i+1}, y_{i+1}, x_{i+2}, y_{i+2}).
\end{aligned}$$

Thus we have defined groups G_i and mappings α_i for all $i = 0, \ldots, 4n-1$. As above, the straightforward verification shows that the mapping α_{4n-1} gives rise to a G-homomorphism $\alpha_{4n-1} : G_S \longrightarrow G_{4n-1}$. We repeat now the above construction once more time with G_{4n-1} in the place of G_0, α_{4n-1} in the place of β, and $\bar{t}_j$ in the place of t_j. We denote the corresponding groups and homomorphisms by $\bar{G}_i$ and $\bar{\alpha}_i : G_S \to \bar{G}_i$.

Put

$$G(U,T) = \bar{G}_{4n-1}, \quad \lambda_\beta = \bar{\alpha}_{4n-1},$$

By induction we have constructed a G-homomorphism

$$\lambda_\beta : G_S \longrightarrow G(U,T).$$

Case: $m > 0, n > 0$. In this case we combine the two previous cases together. To this end we take the group H_{m-1} and the homomorphism $\theta_{m-1} : G[X] \to H_{m-1}$ constructed in the first case and put them as the input for the construction in the second case. Namely, put

$$G_0 = \left\langle H_{m-1}, r_m | [C_{H_{m-1}}(c_m^{z_m^{\theta_{m-1}}} x_1^{-\theta_{m-1}}), r_m] = 1 \right\rangle,$$

and define the homomorphism α_0 as follows

$$z_m^{\alpha_0} = z_m^{\theta_{m-1}} r_m, \quad x_1^{\alpha_0} = a_1^{r_m}, \quad y_1^{\alpha_0} = r_m^{-1} b_1, \quad s^{\alpha_0} = s^{\theta_{m-1}} \quad (s \in X, s \neq z_m, x_1, y_1).$$

Now we apply the construction from the second case. Thus we have defined groups G_i and mappings $\alpha_i : G[X] \to G_i$ for all $i = 0, \ldots, 4n-1$. As above, the straightforward verification shows that the mapping α_{4n-1} gives rise to a G-homomorphism $\alpha_{4n-1} : G_S \longrightarrow G_{4n-1}$.

We repeat now the above construction once more time with G_{4n-1} in place of G_0 and α_{4n-1} in place of β. This results in a group $\bar{G}_{4n-1}$ and a homomorphism $\bar{\alpha}_{4n-1} : G_S \to \bar{G}_{4n-1}$.

Put

$$G(U,T) = \bar{G}_{4n-1}, \quad \lambda_\beta = \bar{\alpha}_{4n-1}.$$

We have constructed a G-homomorphism

$$\lambda_\beta : G_S \longrightarrow G(U,T).$$

We proved the proposition for all three types of equations (15), (16), (17), as required. $\square$

PROPOSITION 5.2. *Let $S = 1$ be a regular quadratic equation* (2) *and*

$$\beta : G_{R(S)} \to G$$

a solution of $S = 1$ in G in a general position. Then the homomorphism

$$\lambda_\beta : G_{R(S)} \to G(U, T)$$

is a monomorphism.

PROOF. In the proof of this proposition we use induction on the atomic rank of the equation in the same way as in the proof of Theorem 1 in [**12**].

Since all the intermediate groups are also fully residually free by induction it suffices to prove the following:

1. $n = 1$, $m = 0$; prove that $\psi = \alpha_3$ is an embedding of G_S into G_3.
2. $n = 2$, $m = 0$; prove that $\psi = \alpha_4$ is a monomorphism on $H = \langle G, x_1, y_1 \rangle$.
3. $n = 1$, $m = 1$; prove that $\psi = \alpha_3 \bar{\alpha}_0$ is a monomorphism on $H = \langle G, z_1 \rangle$.
4. $n = 0$, $m > 2$; prove that $\theta_2 \bar{\theta}_2$ is an embedding of G_S into $\bar{H}_2$.

Now we consider all these cases one by one.

Case 1. Choose an arbitrary nontrivial element $h \in G_S$. It can be written in the form

$$h = g_1 \; v_1(x_1, y_1) \; g_2 \; v_2(x_1, y_1) \; g_3 \ldots v_n(x_1, y_1) \; g_{n+1},$$

where $1 \neq v_i(x_1, y_1) \in F(x_1, y_1)$ are words in x_1, y_1, not belonging to the subgroup $\langle [x_1, y_1] \rangle$, and $1 \neq g_i \in G, g_i \notin \langle [a, b] \rangle$ (with the exception of g_1 and g_{n+1}, they could be trivial). Then

$$h^\psi = g_1 \; v_1(t_3 t_1 a, t_2 b) \; g_2 \; v_2(t_3 t_1 a, t_2 b) \; g_3 \cdots v_n(t_3 t_1 a, t_2 b) \; g_{n+1}. \tag{28}$$

The group $G(U, T)$ is obtained from G by three HNN-extensions (extensions of centralizers), so every element in $G(U, T)$ can be rewritten to its reduced form by making finitely many pinches. It is easy to see that the leftmost occurrence of either t_3 or t_1 in the product (28) occurs in the reduced form of h^ψ uncancelled.

Case 2. $x_1 \to t_4^{-1} t_2 a_1, \; y_1 \to t_4^{-1} t_3 t_1 b_1 t_4, \; x_2 \to t_4^{-1} a_2 t_4, \; y_2 \to t_4^{-1} b_2$. Choose an arbitrary nontrivial element $h \in H = G * F(x_1, y_1)$. It can be written in the form

$$h = g_1 \; v_1(x_1, y_1) \; g_2 \; v_2(x_1, y_1) \; g_3 \ldots v_n(x_1, y_1) \; g_{n+1},$$

where $1 \neq v_i(x_1, y_1) \in F(x_1, y_1)$ are words in x_1, y_1, and $1 \neq g_i \in G$ (with the exception of g_1 and g_{n+1}, they could be trivial). Then

$$h^\psi = g_1 \; v_1(t_4^{-1} t_2 a, (t_3 t_1 b)^{t_4}) \; g_2 \; v_2(t_4^{-1} t_2 a, (t_3 t_1 b)^{t_4}) \; g_3 \cdots v_n(t_4^{-1} t_2 a, (t_3 t_1 b)^{t_4}) \; g_{n+1}. \tag{29}$$

The group $G(U, T)$ is obtained from G by four HNN-extensions (extensions of centralizers), so every element in $G(U, T)$ can be rewritten to its reduced form by making finitely many pinches. It is easy to see that the leftmost occurrence of either t_4 or t_1 in the product (29) occurs in the reduced form of h^ψ uncancelled.

Case 3. We have an equation $c^z [x, y] = c[a, b]$, $z \to z r_1 \bar{r}_1, \; x \to (t_2 a^{r_1})^{\bar{r}_1}, \; y \to \bar{r}_1^{-1} t_3 t_1 r_1^{-1} b$, and $[r_1, ca^{-1}] = 1, \; [\bar{r}_1, (c^{r_1} a^{-r_1} t_2^{-1})] = 1$. Here we can always suppose, that $[c, a] \neq 1$, by changing a solution, hence $[r_1, \bar{r}_1] \neq 1$. The proof for this case is a repetition of the proof of Proposition 11 in [**12**].

Case 4. We will consider the case when $m = 3$; the general case can be considered similarly. We have an equation $c_1^{z_1}c_2^{z_2}c_3^{z_3} = c_1c_2c_3$, and can suppose $[c_i, c_{i+1}] \neq 1$.

We will prove that $\psi = \theta_2\bar{\theta}_1$ is an embedding. The images of z_1, z_2, z_3 under $\theta_2\bar{\theta}_1$ are the following:

$$z_1 \to c_1r_1\bar{r}_1,\ z_2 \to c_2r_1r_2\bar{r}_1,\ z_3 \to c_3r_2,$$

where

$$[r_1, c_1c_2] = 1,\ [r_2, c_2^{r_1}c_3] = 1,\ [\bar{r}_1, c_1^{r_1}c_2^{r_1r_2}] = 1.$$

Let w be a reduced word in $G * F(z_i, i = 1, 2, 3)$, which does not have subwords $c_1^{z_1}$. We will prove that if $w^\psi = 1$ in $\bar{H}_1$, then $w \in N$, where N is the normal closure of the element $c_1^{z_1}c_2^{z_2}c_3^{z_3}c_3^{-1}c_2^{-1}c_1^{-1}$. We use induction on the number of occurrences of $z_1^{\pm 1}$ in w. The induction basis is obvious, because homomorphism ψ is injective on the subgroup $< F, z_2, z_3 >$.

Notice, that the homomorphism ψ is also injective on the subgroup $K =< z_1z_2^{-1}, z_3, F >$.

Consider $\bar{H}_1$ as an HNN-extension by letter $\bar{r}_1$. Suppose $w^\psi = 1$ in $\bar{H}_1$. Letter $\bar{r}_1$ can disappear in two cases: 1) $w \in KN$, 2) there is a pinch between $\bar{r}_1^{-1}$ and $\bar{r}_1$ (or between $\bar{r}_1$ and $\bar{r}_1^{-1}$) in w^ψ. This pinch corresponds to some element $z_{1,2}^{-1}uz'_{1,2}$ (or $z_{1,2}u(z'_{1,2})^{-1}$), where $z_{1,2}, z'_{1,2} \in \{z_1, z_2\}$.

In the first case $w^\psi \neq 1$, because $w \in K$ and $w \notin N$.

In the second case, if the pinch happens in $(z_{1,2}u(z'_{1,2})^{-1})^\psi$, then $z_{1,2}u(z'_{1,2})^{-1} \in KN$, therefore it has to be at least one pinch that corresponds to $(z_{1,2}^{-1}uz'_{1,2})^\psi$. We can suppose, up to a cyclic shift of w, that $z_{1,2}^{-1}$ is the first letter, w does not end with some $z''_{1,2}$, and w cannot be represented as $z_{1,2}^{-1}uz'_{1,2}v_1z''_{1,2}v_2$, such that $z'_{1,2}v_1 \in KN$. A pinch can only happen if $z_{1,2}^{-1}uz'_{1,2} \in< c_1^{z_1}c_2^{z_2} >$. Therefore, either $z_{1,2} = z_1$, or $z'_{1,2} = z_1$, and one can replace $c_1^{z_1}$ by $c_1c_2c_3c_3^{-z_3}c_2^{-z_2}$, therefore replace w by w_1 such that $w = uw_1$, where u is in the normal closure of the element $c_1^{z_1}c_2^{z_2}c_3^{z_3}c_3^{-1}c_2^{-1}c_1^{-1}$, and apply induction. $\square$

The embedding $\lambda_\beta : G_S \to G(U, T)$ allows one to construct effectively discriminating sets of solutions in G of the equation $S = 1$. Indeed, by the construction above the group $G(U, T)$ is union of the following chain of length $2K = 2K(m, n)$ of extension of centralizers:

$$G = H_0 \leqslant H_1 \ldots \leqslant H_{m-1} \leqslant G_0 \leqslant G_1 \leqslant \ldots \leqslant G_{4n-1} =$$

$$= \bar{H}_0 \leqslant \bar{H}_1 \leqslant \ldots \leqslant \bar{H}_{m-1} = \bar{G}_0 \leqslant \ldots \leqslant \bar{G}_{4n-1} = G(U, T).$$

Now, every $2K$-tuple $p \in \mathbb{N}^{2K}$ determines a G-homomorphism

$$\xi_p : G(U, T) \to G.$$

Namely, if Z_i is the i-th term of the chain above then Z_i is an extension of the centralizer of some element $g_i \in Z_{i-1}$ by a stable letter t_i. The G-homomorphism ξ_p is defined as composition

$$\xi_p = \psi_1 \circ \ldots \circ \psi_K$$

of homomorphisms $\psi_i : Z_i \to Z_{i-1}$ which are identical on Z_{i-1} and such that $t_i^{\psi_i} = g_i^{p_i}$, where p_i is the i-th component of p.

It follows (see [**16**, Section 1.4]) that for every unbounded set of tuples $P \subset \mathbb{N}^{2K}$ the set of homomorphisms

$$\Xi_P = \{\xi_p \mid p \in P\}$$

G-discriminates $G(U,T)$ into G. Therefore, (since λ_β is monic), the family of G-homomorphisms

$$\Xi_{P,\beta} = \{\lambda_\beta \xi_p \mid \xi_p \in \Xi_P\}$$

G-discriminates G_S into G.

One can give another description of the set $\Xi_{P,\beta}$ in terms of the basic automorphisms from the basic sequence Γ. Observe first that

$$\lambda_\beta \xi_p = \phi_{2K,p}\beta,$$

therefore

$$\Xi_{P,\beta} = \{\phi_{2K,p}\beta \mid \ p \in P\}.$$

We summarize the discussion above as follows.

THEOREM 5.3. *Let G be a finitely generated fully residually free group, $S = 1$ a regular standard quadratic orientable equation, and Γ its basic sequence of automorphisms. Then for any solution $\beta : G_S \to G$ in general position, any positive integer $J \geq 2$, and any unbounded set $P \subset \mathbb{N}^{JK}$ the set of G-homomorphisms $\Xi_{P,\beta}$ G-discriminates $G_{R(S)}$ into G. Moreover, for any fixed tuple $p' \in \mathbb{N}^{tK}$ the family*

$$\Xi_{P,\beta,p'} = \{\phi_{tK,p'}\theta \mid \theta \in \Xi_{P,\beta}\}$$

G-discriminates $G_{R(S)}$ into G.

For tuples $f = (f_1, \ldots, f_k)$ and $g = (g_1, \ldots, g_m)$ denote the tuple

$$fg = (f_1, \ldots, f_k, g_1, \ldots, g_m).$$

Similarly, for a set of tuples P put

$$fPg = \{fpg \mid p \in P\}.$$

COROLLARY 5.4. *Let G be a finitely generated fully residually free group, $S = 1$ a regular standard quadratic orientable equation, Γ the basic sequence of automorphisms of S, and $\beta : G_S \to G$ a solution of $S = 1$ in general position. Suppose $P \subseteq \mathbb{N}^{2K}$ is unbounded set, and $f \in \mathbb{N}^{Ks}$, $g \in \mathbb{N}^{Kr}$ for some $r, s \in \mathbb{N}$. Then there exists a number N such that if f is N-large and $s \geq 2$ then the family*

$$\Phi_{P,\beta,f,g} = \{\phi_{K(r+s+2),q}\beta \mid q \in fPg\}$$

G-discriminates $G_{R(S)}$ into G.

PROOF. By Theorem 5.3 it suffices to show that if f is N-large for some N then $\beta_f = \phi_{2K,f}\beta$ is a solution of $S = 1$ in general position, i.e., the images of some particular finitely many non-commuting elements from $G_{R(S)}$ do not commute in G. It has been shown above that the set of solutions $\{\phi_{2K,h}\beta \mid h \in \mathbb{N}^{2K}\}$ is a discriminating set for $G_{R(S)}$. Moreover, for any finite set M of non-trivial elements from $G_{R(S)}$ there exists a number N such that for any N-large tuple $h \in \mathbb{N}^{2K}$ the solution $\phi_{2K,h}\beta$ discriminates all elements from M into G. Hence the result. $\square$

6. Small cancellation solutions of standard orientable equations

Let $S(X) = 1$ be a standard regular orientable quadratic equation over F written in the form (18):

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] = \prod_{i=1}^{m} e_i^{-1} c_i e_i \prod_{i=1}^{n} [a_i, b_i].$$

In this section we construct solutions in F of $S(X) = 1$ which satisfy some small cancellation conditions.

Definition 6.1. Let $S = 1$ be a standard regular orientable quadratic equation written in the form (18). We say that a solution $\beta : F_S \to F$ of $S = 1$ satisfies the small cancellation condition $(1/\lambda)$ with respect to the set $\bar{\mathcal{W}}_{\Gamma,L}$ if the following conditions are satisfied:

1) β is in general position;
2) for any 2-letter word $uv \in \mathcal{W}_{\Gamma,L}$ (in the alphabet Y) cancellation in the word $u^\beta v^\beta$ does not exceed $(1/\lambda)\min\{|u^\beta|, |v^\beta|\}$ (we assume here and below that u^β, v^β are given by their reduced forms in F);
3) cancellation in a word $u^\beta v^\beta$ does not exceed $(1/\lambda)\min\{|u^\beta|, |v^\beta|\}$ provided u, v satisfy one of the conditions below:
   a) $u = z_i, v = (z_{i-1}^{-1} c_{i-1}^{-1} z_{i-1})$,
   b) $u = c_i, v = z_i$,
   c) $u = v = c_i$.

Notation 6.2. For a homomorphism $\beta : F[X] \to F$ by C_β we denote the set of all elements that cancel in $u^\beta v^\beta$ where u, v are as in 2), 3) from Definition 6.1.

Lemma 6.3. *Let u, v be cyclically reduced elements of $G*H$ such that $|u|, |v| \geqslant 2$. If for some $m, n > 1$ elements u^m and v^n have a common initial segment of length $|u|+|v|$, then u and v are both powers of the same element $w \in G*H$. In particular, if both u and v are not proper powers then $u = v$.*

Proof. The same argument as in the case of free groups.

Corollary 6.4. *If $u, v \in F$, $[u, v] \neq 1$, then for any $\lambda \geqslant 0$ there exist m_0, n_0 such that for any $m \geqslant m_0, n \geqslant n_0$ cancellation between u^m and v^n is less than $\frac{1}{\lambda}\max\{|u^m|, |v^n|\}$.*

Lemma 6.5. *Let $S(X) = 1$ be a standard regular orientable quadratic equation written in the form (18):*

$$\prod_{i=1}^{m} z_i^{-1} c_i z_i \prod_{i=1}^{n} [x_i, y_i] = \prod_{i=1}^{m} e_i^{-1} c_i e_i \prod_{i=1}^{n} [a_i, b_i], \quad n \geqslant 1,$$

where all c_i are cyclically reduced, and

$$\beta_1 : x_i \to a_i, y_i \to b_i, z_i \to e_i$$

a solution of $S = 1$ in F in general position. Then for any $\lambda \in \mathbb{N}$ there are positive integers m_i, n_i, k_i, q_j and a tuple $p = (p_1, \ldots p_m)$ such that the map $\beta : F[X] \to F$ defined by

$$x_1^\beta = (\tilde{b}_1^{n_1} \tilde{a}_1)^{[\tilde{a}_1, \tilde{b}_1]^{m_1}}, \quad y_1^\beta = ((\tilde{b}_1^{n_1} \tilde{a}_1)^{k_1} \tilde{b}_1)^{[\tilde{a}_1, \tilde{b}_1]^{m_1}}, \quad \textit{where } \tilde{a}_1 = x_1^{\phi_m \beta_1}, \ \tilde{b}_1 = y_1^{\phi_m \beta_1}$$

$$x_i^\beta = (b_i^{n_i} a_i)^{[a_i,b_i]^{m_i}}, \quad y_i^\beta = ((b_i^{n_i} a_i)^{k_i} b_i)^{[a_i,b_i]^{m_i}}, \quad i = 2, \dots n,$$

$$z_i^\beta = c_i^{q_i} z_i^{\phi_m \beta_1}, \quad i = 1, \dots m,$$

is a solution of $S = 1$ satisfying the small cancellation condition $(1/\lambda)$ with respect to $\bar{\mathcal{W}}_{\Gamma,L}$. Moreover, one can choose the solution β_1 such that if $u = c_i^{z_i}$ or $u = x_j^{-1}$ and $v = c_1^{z_1}$, then the cancellation between u^β and v^β is less than $(1/\lambda)\min\{|u|, |v|\}$.

PROOF. The solution

$$x_i \to a_i, y_i \to b_i, z_i \to e_i$$

$i = 1, \dots, n, j = 1, \dots, m$ is in general position, therefore the neighboring items in the sequence

$$c_1^{e_1}, \dots, c_m^{e_m}, [a_1, b_1], \dots, [a_n, b_n]$$

do not commute.

There is a homomorphism $\theta_{\beta_1} : F_S \to \bar{F} = F(\bar{U}, \bar{T})$ into the group $\bar{F}$ obtained from F by a series of extensions of centralizers, such that $\beta = \theta_{\beta_1}\psi_p$, where $\psi_p : \bar{F} \to F$. This homomorphism θ_{β_1} is a monomorphism on $F * F(z_1, \dots, z_m)$ (this follows from the proof of Theorem 4 in [**12**], where the same sequence of extensions of centralizers is constructed).

The set of solutions ψ_p for different tuples p and numbers m_i, n_i, k_i, q_j is a discriminating family for $\bar{F}$. We just have to show that the small cancellation condition for β is equivalent to a finite number of inequalities in the group $\bar{F}$.

We have $z_i^\beta = c_i^{q_i} z_i^{\phi_m \beta_1}$ such that $\beta_1(z_i) = e_i$, and $p = (p_1, \dots, p_m)$ is a large tuple. Denote $\bar{A}_j = A_j^{\beta_1}, \ j = 1, \dots, m$. Then it follows from Lemma 4.6 that

$z_i^\beta = c_i^{q_i+1} e_i \bar{A}_{i-1}^{p_{i-1}} c_{i+1}^{e_{i+1}} \bar{A}_i^{p_i - 1}$, where $i = 2, \dots, m-1$

$z_m^\beta = c_m^{q_m+1} e_m \bar{A}_{m-1}^{p_{m-1}} a_1^{-1} \bar{A}_m^{p_m - 1}$,

where

$$\begin{aligned}
\bar{A}_1 &= c_1^{e_1} c_2^{e_2}, \\
\bar{A}_2 &= \bar{A}_1(p_1) = \bar{A}_1^{-p_1} c_2^{e_2} \bar{A}_1^{p_1} c_3^{e_3}, \\
&\vdots \\
\bar{A}_i &= \bar{A}_{i-1}^{-p_{i-1}} c_i^{e_i} \bar{A}_{i-1}^{p_{i-1}} c_{i+1}^{e_{i+1}}, \ i = 2, \dots, m-1, \\
\bar{A}_m &= \bar{A}_{m-1}^{-p_{m-1}} c^{e_m} \bar{A}_{m-1}^{p_{m-1}} a_1^{-1}.
\end{aligned}$$

One can choose p such that $[\bar{A}_i, \bar{A}_{i+1}] \neq 1, [\bar{A}_{i-1}, c_{i+1}^{e_{i+1}}] \neq 1, [\bar{A}_{i-1}, c_i^{e_i}] \neq 1$ and $[\bar{A}_m, [a_1, b_1]] \neq 1$, because their pre-images do not commute in $\bar{F}$. We need the second and third inequality here to make sure that $\bar{A}_i$ does not end with a power of $\bar{A}_{i-1}$. Alternatively, one can prove by induction on i that p can be chosen to satisfy these inequalities. Then $c_i^{z_i^\beta}$ and $c_{i+1}^{z_{i+1}^\beta}$ have small cancellation, and $c_m^{z_m^\beta}$ has small cancellation with $x_1^{\pm\beta}, y_1^{\pm\beta}$.

Let

$$x_i^\beta = (b_i^{n_i} a_i)^{[a_i,b_i]^{m_i}}, \quad y_i^\beta = ((b_i^{n_i} a_i)^{k_i} b_i)^{[a_i,b_i]^{m_i}}, \quad i = 2, \dots, n$$

for some positive integers m_i, n_i, k_i, s_j which values we will specify in a due course. Let $uv \in \bar{\mathcal{W}}_\Gamma$. There are several cases to consider.

1) $uv = x_i x_i$. Then

$$u^\beta v^\beta = (b_i^{n_i} a_i)^{[a_i,b_i]^{m_i}} (b_i^{n_i} a_i)^{[a_i,b_i]^{m_i}}.$$

Observe that the cancellation between $(b_i^{n_i} a_i)$ and $(b_i^{n_i} a_i)$ is not more then $|a_i|$. Hence the cancellation in $u^\beta v^\beta$ is not more then $|[a_i,b_i]^{m_i}|+|a_i|$. We chose $n_i \gg m_i$ such that

$$|[a_i,b_i]^{m_i}| + |a_i| < (1/\lambda)|(b_i^{n_i} a_i)^{[a_i,b_i]^{m_i}}|$$

which is obviously possible. Similar arguments prove the cases $uv = x_i y_i$ and $uv = y_i x_i$.

2) In all other cases the cancellation in $u^\beta v^\beta$ does not exceed the cancellation between $[a_i,b_i]^{m_i}$ and $[a_{i+1},b_{i+1}]^{m_{i+1}}$, hence by Lemma 6.3 it is not greater than $|[a_i,b_i]| + |[a_{i+1},b_{i+1}]|$.

Let $u = z_i^\beta, v = c_{i-1}^{-z_{i-1}^\beta}$. The cancellation is the same as between $\bar{A}_{2i}^{p_{2i}}$ and $\bar{A}_{i-1}^{-p_{i-1}}$ and, therefore, small.

Since c_i is cyclically reduced, there is no cancellation between c_i and z_i^β.

The first statement of the lemma is proved.

We now will prove the second statement of the lemma. We can choose the initial solution $e_1,\dots,e_m,a_1,b_1,\dots,a_n,b_n$ so that $[c_1^{e_1}c_2^{e_2}, c_3^{e_3}\dots c_i^{e_i}] \neq 1$ ($i \geqslant 3$), $[c_1^{e_1}c_2^{e_2},[a_i,b_i]] \neq 1, (i=2,\dots,n)$ and $[c_1^{e_1}c_2^{e_2}, b_1^{-1}a_1^{-1}b_1] \neq 1$. Indeed, the equations $[c_1^{z_1}c_2^{z_2}, c_3^{z_3}\dots c_i^{z_i}] = 1$, $[c_1^{z_1}c_2^{z_2},[x_i,y_i]] = 1, (i=2,\dots,n)$ and $[c_1^{z_1}c_2^{z_2}, y_1^{-1}x_1^{-1}y_1] = 1$ are not consequences of the equation $S=1$, and, therefore, there is a solution of $S(X)=1$ which does not satisfy any of these equations.

To show that $u = c_i^{z_i^\beta}$ and $v = c_1^{z_1^\beta}$, have small cancellation, we have to show that p can be chosen so that $[\bar{A}_1,\bar{A}_i] \neq 1$ (which is obvious, because the pre-images in $\bar{G}$ do not commute), and that $\bar{A}_i^{-1}$ does not begin with a power of $\bar{A}_1$. The period $\bar{A}_i^{-1}$ has form $(c_{i+1}^{-z_{i+1}}\dots c_3^{-z_3}\bar{A}_1^{-p_2}\dots)$. It begins with a power of $\bar{A}_1$ if and only if $[\bar{A}_1, c_3^{e_3}\dots c_i^{e_i}] = 1$, but this equality does not hold.

Similarly one can show, that the cancellation between $u = x_j^{-\beta}$ and $v = c_1^{z_1^\beta}$ is small. □

Lemma 6.6. *Let $S(X)=1$ be a standard regular orientable quadratic equation of the type* (17)

$$\prod_{i=1}^m z_i^{-1} c_i z_i = c_1^{e_1}\dots c_m^{e_m} = d,$$

where all c_i are cyclically reduced, and

$$\beta_1 : z_i \to e_i$$

a solution of $S=1$ in F in general position. Then for any $\lambda \in \mathbb{N}$ there is a positive integer s and a tuple $p=(p_1,\dots p_K)$ such that the map $\beta : F[X] \to F$ defined by

$$z_i^\beta = c_i^{q_i} z_i^{\phi_K \beta_1} d^s,$$

is a solution of $S=1$ satisfying the small cancellation condition $(1/\lambda)$ *with respect to $\bar{\mathcal{W}}_{\Gamma,L}$ with one exception when $u=d$ and $v = c_{m-1}^{-z_{m-1}}$ (in this case d cancels out in v^β). Notice, however, that such word uv occurs only in the product wuv with $w = c_2^{z_2}$, in which case cancellation between w^β and dv^β is less than* $\min\{|w^\beta|,|dv^\beta|\}$.

PROOF. Solution β is chosen the same way as in the previous lemma (except for the multiplication by d^s) on the elements $z_i,\ i \neq m$. We do not take s very large, we just need it to avoid cancellation between z_2^β and d. Therefore the cancellation between $c_i^{z_i^\beta}$ and $c_{i+1}^{\pm z_{i+1}^\beta}$ is small for $i < m-1$. Similarly, for $u = c_2^{z_2},\ v = d,\ w = c_{m-1}^{-z_{m-1}}$, we can make the cancellation between u^β and dw^β less than $\min\{|u^\beta|, |dw^\beta|\}$. $\square$

LEMMA 6.7. *Let $U, V \in \bar{\mathcal{W}}_{\Gamma,L}$ such that $UV = U \circ V$ and $UV \in \bar{\mathcal{W}}_{\Gamma,L}$.*

1. Let $n \neq 0$. If u is the last letter of U and v is the first letter of V then cancellation between U^β and V^β is equal to the cancellation between u^β and v^β.

2. Let $n = 0$. If u_1u_2 are the last two letters of U and v_1, v_2 are the first two letters of V then cancellation between U^β and V^β is equal to the cancellation between $(u_1u_2)^\beta$ and $(v_1v_2)^\beta$.

PROOF. Since β has the small cancellation property with respect to $\bar{\mathcal{W}}_{\Gamma,L}$, this implies that the cancellation in $U^\beta V^\beta$ is equal to the cancellation in $u^\beta v^\beta$, which is equal to some element in C_β. This proves the lemma. $\square$

Let $w \in \bar{\mathcal{W}}_{\Gamma,L}, W = w^{\phi_j}$. We start with the canonical N-large A-representation of W:

$$W = B_1 \circ A^{q_1} \circ \cdots \circ B_k \circ A^{q_k} \circ B_{k+1} \tag{30}$$

where $|q_i| \geqslant N$ and $\max_j(B_i) \leqslant r$.

Since the occurrences A^{q_i} above are stable we have

$$B_1 = \bar{B}_1 \circ A^{sgn(q_1)},\ \ B_i = A^{sgn(q_{i-1})} \circ \bar{B}_i \circ A^{sgn(q_i)}\ (2 \leqslant i \leqslant k),\ \ B_{k+1} = A^{sgn(q_k)} \circ \bar{B}_{k+1}.$$

Denote $A^\beta = c^{-1}A'c$, where A' is cyclically reduced, and $c \in C_\beta$. Then

$$B_1^\beta = \bar{B}_1^\beta c^{-1}(A')^{sgn(q_1)}c,\ \ B_i^\beta = c^{-1}(A')^{sgn(q_{i-1})}c\bar{B}_i^\beta c^{-1}(A')^{sgn(q_i)}c,$$

$$B_{k+1}^\beta = c^{-1}(A')^{sgn(q_k)}c\bar{B}_{k+1}^\beta.$$

By Lemma 6.7 we can assume that the cancellation in the words above is small, i.e., it does not exceed a fixed number σ which is the maximum length of words from C_β. To get an N-large canonical A'-decomposition of W^β one has to take into account stable occurrences of A'. To this end, put $\varepsilon_i = 0$ if ${A'}^{sgn(q_i)}$ occurs in the reduced form of $\bar{B}_i^\beta c^{-1}(A')^{sgn(q_i)}$ as written (the cancellation does not touch it), and put $\varepsilon_i = sgn(q_i)$ otherwise. Similarly, put $\delta_i = 0$ if ${A'}^{sgn(q_i)}$ occurs in the reduced form of $(A')^{sgn(q_i)}c\bar{B}_{i+1}^\beta$ as written, and put $\delta_i = sgn(q_i)$ otherwise.

Now one can rewrite W^β in the following form

$$W^\beta = E_1 \circ (A')^{q_1-\varepsilon_1-\delta_1} \circ E_2 \circ (A')^{q_2-\varepsilon_2-\delta_2} \circ \cdots \circ (A')^{q_k-\varepsilon_k-\delta_k} \circ E_{k+1}, \tag{31}$$

where $E_1 = (B_1^\beta c^{-1}(A')^{\varepsilon_1}),\ E_2 = ((A')^{\delta_1}cB_2^\beta c^{-1}(A')^{\varepsilon_2}),\ E_{k+1} = ((A')^{\delta_k}cB_{k+1}^\beta)$.

Observe, that d_i and ε_i, δ_i can be effectively computed from W and β. It follows that one can effectively rewrite W^β in the form (31) and the form is unique.

The decomposition (31) of W^β induces a corresponding A^*-decomposition of W. Namely, if the canonical N-large A^*-decomposition of W has the form:

$$D_1(A^*)^{q_1}D_2 \cdots D_k(A^*)^{q_k}D_{k+1}$$

then the induced one has the form: $W =$

(32)
$$(D_1 A^{*\varepsilon_1}) A^{*q_1-\varepsilon_1-\delta_1} (A^{*\delta_1} D_2 A^{*\varepsilon_2}) \cdots (A^{*\delta_{k-1}} D_k A^{*\varepsilon_k}) A^{*q_k-\varepsilon_k-\delta_k} (A^{*\delta_k} D_{k+1}).$$

We call this decomposition the *induced* A^*-decomposition of W with respect to β and write it in the form:

$$W = D_1^* (A^*)^{q_1^*} D_2^* \cdots D_k^* (A^*)^{q_k^*} D_{k+1}^*, \tag{33}$$

where $D_i^* = (A^*)^{\delta_{i-1}} D_i (A^*)^{\varepsilon_i}$, $q_i^* = q_i - \varepsilon_i - \delta_i$, and, for uniformity, $\delta_1 = 0$ and $\varepsilon_{k+1} = 0$.

Lemma 6.8. *For given positive integers j, M, N there is a constant $C = C(j, M, N) > 0$ such that if $p_{t+1} - p_t > C$ for every $t = 1, \ldots, j-1$, and a word $W \in \bar{\mathcal{W}}_{\Gamma,L}$ has a canonical N-large A^*-decomposition (33), then this decomposition satisfies the following conditions:*

(34)
$$(D_1^*)^\beta = E_1 \circ_\theta (cR^\beta), \;\; (D_i^*)^\beta = (R^{-\beta} c^{-1}) \circ_\theta E_i \circ_\theta (cR^\beta), \;\; (D_{k+1}^*)^\beta = (R^{-\beta} c^{-1}) \circ_\theta E_{k+1},$$

where $\theta < |A| - M$. Moreover, this constant C can be found effectively.

Proof. Applying homomorphism β to the reduced A^*-decomposition of W_σ (33) we can see that

$$W_\sigma^\beta = \left((D_1^*)^\beta R^\beta c\right) (A')^{q_1^*} \left(cR^\beta (D_2^*)^\beta R^{-\beta} c^{-1}\right) (A')^{q_2^*} \ldots$$
$$\left(cR^\beta (D_k^*)^\beta R^{-\beta} c^{-1}\right) (A')^{q_k^*} \left(cR^\beta (D_{k+1}^*)^\beta\right).$$

Observe that this decomposition has the same powers of A' as the canonical N-large A'-decomposition (31). From the uniqueness of such decompositions we deduce that

$$E_1 = (D_1^*)^\beta R^\beta c, \quad E_i = cR^\beta (D_i^*)^\beta R^{-\beta} c^{-1}, \quad E_{k+1} = cR^\beta (D_{k+1}^*)^\beta$$

Rewriting these equalities one can get

$$(D_1^*)^\beta = E_1 \circ_\theta (cR^\beta), \;\; (D_i^*)^\beta = (R^{-\beta} c^{-1}) \circ_\theta E_i \circ_\theta (cR^\beta), \;\; (D_{k+1}^*)^\beta = (R^{-\beta} c^{-1}) \circ_\theta E_{k+1}$$

and $\theta \ll |A|$. Indeed, in the decomposition (31) every occurrence $(A')^{q_i-\varepsilon_i-\delta_i}$ is stable hence E_i starts (ends) on A'. The N-large rank of R is at most $rank_N(A)$, and β has small cancellation. Taking $p_{j+1} \gg p_j$ we may assume that $|A'| \gg |c|, |R^\beta|$. □

Notice, that one can effectively write down the induced A^*-decomposition of W with respect to β.

We summarize the discussion above in the following statement.

Lemma 6.9. *For given positive integers j, N there is a constant $C = C(j, N)$ such that if $p_{t+1} - p_t > C$, for every $t = 1, \ldots, j-1$, then for any $W \in \bar{\mathcal{W}}_{\Gamma,L}$ the following conditions are equivalent:*

1. *Decomposition* (30) *is the canonical (the canonical N-large) A-decomposition of W,*
2. *Decomposition* (31) *is the canonical (the canonical N-large) A'-decomposition of W^β,*
3. *Decomposition* (32) *is the canonical (the canonical N-large) A^*-decomposition of W.*

7. Implicit function theorem for quadratic equations

In this section we prove Theorem A for orientable quadratic equations over a free group $F = F(A)$. Namely, we prove the following statement.

Let $S(X, A) = 1$ be a regular standard orientable quadratic equation over F. Then every equation $T(X, Y, A) = 1$ compatible with $S(X, A) = 1$ admits an effective complete S-lift.

A special discriminating set of solutions $\mathcal{L}$ and the corresponding cut equation Π.

Below we continue to use notations from the previous sections. Fix a solution β of $S(X, A) = 1$ which satisfies the cancellation condition $(1/\lambda)$ (with $\lambda > 10$) with respect to $\bar{\mathcal{W}}_\Gamma$.

Put

$$x_i^\beta = \tilde{a}_i, y_i^\beta = \tilde{b}_i, z_i^\beta = \tilde{c}_i.$$

Recall that

$$\phi_{j,p} = \gamma_j^{p_j} \cdots \gamma_1^{p_1} = \overleftarrow{\Gamma}_j^{p}$$

where $j \in \mathbb{N}$, $\Gamma_j = (\gamma_1, \ldots, \gamma_j)$ is the initial subsequence of length j of the sequence $\Gamma^{(\infty)}$, and $p = (p_1, \ldots, p_j) \in \mathbb{N}^j$. Denote by $\psi_{j,p}$ the following solution of $S(X) = 1$:

$$\psi_{j,p} = \phi_{j,p}\beta.$$

Sometimes we omit p in $\phi_{j,p}, \psi_{j,p}$ and simply write ϕ_j, ψ_j.

Below we continue to use notation:

$$A = A_j = A_j, \ A^* = A_j^* = A^*(\phi_j) = R_j^{-1} \circ A_j \circ R_j, \ d = d_j = |R_j|.$$

Recall that R_j has rank $\leq j - K + 2$ (Lemma 4.33). By A' we denote the cyclically reduced form of A^β (hence of $(A^*)^\beta$). Recall that C_β is the finite set of all initial and terminal segments of elements in $(X^{\pm 1})^\beta$.

Let

$$\Phi = \{\phi_{j,p} \mid j \in \mathbb{N}, p \in \mathbb{N}^j\}.$$

For an arbitrary subset $\mathcal{L}$ of Φ denote

$$\mathcal{L}^\beta = \{\phi\beta \mid \phi \in \mathcal{L}\}.$$

Specifying step by step various subsets of Φ we will eventually ensure a very particular choice of a set of solutions of $S(X) = 1$ in F.

Let $K = K(m, n)$ and $J \in \mathbb{N}, J \geq 3$, a sufficiently large positive integer which will be specified precisely in due course. Put $L = JK$ and define $\mathcal{P}_1 = \mathbb{N}^L$,

$$\mathcal{L}_1 = \{\phi_{L,p} \mid p \in \mathcal{P}_1\}.$$

By Theorem 5.3 the set ${\mathcal{L}_1}^\beta$ is a discriminating set of solutions of $S(X) = 1$ in F. In fact, one can replace the set $\mathcal{P}_1$ in the definition of $\mathcal{L}_1$ by any unbounded subset $\mathcal{P}_2 \subseteq \mathcal{P}_1$, so that the new set is still discriminating. Now we construct by induction a very particular unbounded subset $\mathcal{P}_2 \subseteq \mathbb{N}^L$. Let $a \in \mathbb{N}$ be a natural number and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ a function. Define a tuple

$$p^{(0)} = (p_1^{(0)}, \ldots, p_L^{(0)})$$

where

$$p_1^{(0)} = a, \quad p_{j+1}^{(0)} = p_j^{(0)} + h(0, j).$$

Similarly, if a tuple $p^{(i)} = (p_1^{(i)}, \dots, p_L^{(i)})$ is defined then put $p^{(i+1)} = (p_1^{(i+1)}, \dots, p_L^{(i+1)})$, where

$$p_1^{(i+1)} = p_1^{(i)} + h(i+1, 0), \quad p_{j+1}^{(i+1)} = p_j^{(i+1)} + h(i+1, j).$$

This defines by induction an infinite set

$$\mathcal{P}_{a,h} = \{p^{(i)} \mid i \in \mathbb{N}\} \subseteq \mathbb{N}^L$$

such that any infinite subset of $\mathcal{P}_{f,h}$ is also unbounded.

From now on fix a recursive non-negative monotonically increasing with respect to both variables function h (which will be specified in due course) and put

$$\mathcal{P}_2 = \mathcal{P}_{a,h}, \quad \mathcal{L}_2 = \{\phi_{L,p} \mid p \in \mathcal{P}_2\}.$$

Proposition 7.1. *Let $r \geq 2$ and $K(r+2) \leq L$ then there exists a number a_0 such that if $a \geq a_0$ and the function h satisfies the condition*

$$h(i+1, j) > h(i, j) \quad \text{for any } j = Kr+1, \dots, K(r+2), i = 1, 2, \dots; \tag{35}$$

then for any infinite subset $\mathcal{P} \subseteq \mathcal{P}_2$ the set of solutions

$$\mathcal{L}_{\mathcal{P}}{}^{\beta} = \{\phi_{L,p}\beta \mid p \in \mathcal{P}\}$$

is a discriminating set of solutions of $S(X, A) = 1$.

Proof. The result follows from Corollary 5.4. □

Let $\psi = \psi_p \in \mathcal{L}_2^{\beta}$. Denote by U_ψ the solution X^ψ of the equation $S(X) = 1$ in F. Since $T(X, Y) = 1$ is compatible with $S(X) = 1$ in F the equation $T(U_\psi, Y) = 1$ (in variables Y) has a solution in F, say $Y = V_\psi$. Set

$$\Lambda = \{(U_\psi, V_\psi) \mid \psi \in \mathcal{L}_2^{\beta}\}.$$

It follows that every pair $(U_\psi, V_\psi) \in \Lambda$ gives a solution of the system

$$R(X, Y) = (S(X) = 1 \ \wedge \ T(X, Y) = 1).$$

By Theorem 3.4 there exists a finite set $\mathcal{C}E(R)$ of cut equations which describes all solutions of $R(X, Y) = 1$ in F, therefore there exists a cut equation $\Pi_{\mathcal{L}_3,\Lambda} \in \mathcal{C}E(R)$ and an infinite subset $\mathcal{L}_3 \subseteq \mathcal{L}_2$ such that $\Pi_{\mathcal{L}_3,\Lambda}$ describes all solutions of the type (U_ψ, V_ψ), where $\psi \in \mathcal{L}_3$. We state the precise formulation of this result in the following proposition which, as we have mentioned already, follows from Theorem 3.4.

Proposition 7.2. *Let $\mathcal{L}_2$ and Λ be as above. Then there exists an infinite subset $\mathcal{P}_3 \subseteq \mathcal{P}_2$ and the corresponding set $\mathcal{L}_3 = \{\phi_{L,p} \mid p \in \mathcal{P}_3\} \subseteq \mathcal{L}_2$, a cut equation $\Pi_{\mathcal{L}_3,\Lambda} = (\mathcal{E}, f_X, f_M) \in \mathcal{C}E(R)$, and a tuple of words $Q(M)$ such that the following conditions hold:*

1) *$f_X(\mathcal{E}) \subset X^{\pm 1}$;*
2) *for every $\psi \in \mathcal{L}_3^{\beta}$ there exists a tuple of words $P_\psi = P_\psi(M)$ and a solution $\alpha_\psi : M \to F$ of $\Pi_{\mathcal{L}_3,\Lambda}$ with respect to $\psi : F[X] \to F$ such that:*
   - *the solution $U_\psi = X^\psi$ of $S(X) = 1$ can be presented as $U_\psi = Q(M^{\alpha_\psi})$ and the word $Q(M^{\alpha_\psi})$ is reduced as written,*
   - *$V_\psi = P_\psi(M^{\alpha_\psi})$.*
3) *there exists a tuple of words P such that for any solution (any group solution) (β, α) of $\Pi_{\mathcal{L}_3,\Lambda}$ the pair (U, V), where $U = Q(M^\alpha)$ and $V = P(M^\alpha)$, is a solution of $R(X, Y) = 1$ in F.*

Put

$$\mathcal{P} = \mathcal{P}_3, \quad \mathcal{L} = \mathcal{L}_3, \quad \Pi_{\mathcal{L}} = \Pi_{\mathcal{L}_3,\Lambda}.$$

By Proposition 7.1 the set $\mathcal{L}^\beta$ is a discriminating set of solutions of $S(X) = 1$ in F.

The initial cut equation Π_ϕ.

Now fix a tuple $p \in \mathcal{P}$ and the automorphism $\phi = \phi_{L,p} \in \mathcal{L}$. Recall, that for every $j \leq L$ the automorphism ϕ_j is defined by $\phi_j = \overset{\leftarrow p_j}{\Gamma_j}$, where p_j is the initial subsequence of p of length j. Sometimes we use notation $\psi = \phi\beta, \psi_j = \phi_j\beta$.

Starting with the cut equation $\Pi_{\mathcal{L}}$ we construct a cut equation $\Pi_\phi = (\mathcal{E}, f_{\phi,X}, f_M)$ which is obtained from $\Pi_{\mathcal{L}}$ by replacing the function $f_X : \mathcal{E} \to F[X]$ by a new function $f_{\phi,X} : \mathcal{E} \to F[X]$, where $f_{\phi,X}$ is the composition of f_X and the automorphism ϕ. In other words, if an interval $e \in \mathcal{E}$ in $\Pi_{\mathcal{L}}$ has a label $x \in X^{\pm 1}$ then its label in Π_ϕ is x^ϕ.

Notice, that $\Pi_{\mathcal{L}}$ and Π_ϕ satisfy the following conditions:

a) $\sigma^{f_X\phi\beta} = \sigma^{f_{\phi,X}\beta}$ for every $\sigma \in \mathcal{E}$;
b) the solution of $\Pi_{\mathcal{L}}$ with respect to $\phi\beta$ is also a solution of Π_ϕ with respect to β;
c) any solution (any group solution) of Π_ϕ with respect to β is a solution (a group solution) of $\Pi_{\mathcal{L}}$ with respect to $\phi\beta$.

The cut equation Π_ϕ has a very particular type. To deal with such cut equations we need the following definitions.

Definition 7.3. Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a cut equation. Then the number

$$length(\Pi) = \max\{|f_M(\sigma)| \mid \sigma \in \mathcal{E}\}$$

is called the length of Π. We denote it by $length(\Pi)$ or simply by N_Π.

Notice, by construction, $length(\Pi_\phi) = length(\Pi_{\phi'})$ for every $\phi, \phi' \in \mathcal{L}$. Denote

$$N_{\mathcal{L}} = length(\Pi_\phi).$$

Definition 7.4. A cut equation $\Pi = (\mathcal{E}, f_X, f_M)$ is called a Γ-cut equation in *rank* j $(rank(\Pi) = j)$ and size l if it satisfies the following conditions:

1) let $W_\sigma = f_X(\sigma)$ for $\sigma \in \mathcal{E}$ and $N = (l+2)N_\Pi$. Then for every $\sigma \in \mathcal{E}$ $W_\sigma \in \bar{\mathcal{W}}_{\Gamma,L}$ and one of the following conditions holds:
   1.1) W_σ has N-large rank j and its canonical N-large A_j-decomposition has size $(N, 2)$ i.e., W_σ has the canonical N-large A_j-decomposition
$$W_\sigma = B_1 \circ A_j^{q_1} \circ \ldots B_k \circ A_j^{q_k} \circ B_{k+1}, \tag{36}$$
   with $max_j(B_i) \leq 2$ and $q_i \geq N$;
   1.2) W_σ has rank j and $\max_j(W_\sigma) \leq 2$;
   1.3) W_σ has rank $< j$.
   Moreover, there exists at least one interval $\sigma \in \mathcal{E}$ satisfying the condition 1.1).
2) there exists a solution $\alpha : F[M] \to F$ of the cut equation Π with respect to the homomorphism $\beta : F[X] \to F$.

Lemma 7.5. *Let $l \geq 3$. The cut equation Π_ϕ is a Γ-cut equation in rank L and size l, provided*

$$p_L \geq (l+2)N_{\Pi_\phi} + 3.$$

Proof. By construction the labels of intervals from Π_ϕ are precisely the words of the type x^{ϕ_L} and every such word appears as a label. Observe, that $rank(x_i^{\phi_L}) < L$ for every $i, 1 \leq i \leq n$ (Lemma 4.39, 1a). Similarly, $rank(x_i^{\phi_L}) < L$ for every $i < n$ and $rank(y_n^{\phi_L}) = L$ (Lemma 4.39 1b). Also, $rank(z_i^{\phi_L}) < L$ unless $n = 0$ and $i = m$, in the latter case $z_m^{\phi_L}) = L$ (Lemma 4.39 1c and 1d). Now consider the labels $y_n^{\phi_L}$ and $z_m^{\phi_L})$ (in the case $n = 0$) of rank L. Again, it has been shown in Lemma 4.39 1) that these labels have N-large A_L-decompositions of size $(N, 2)$, as required in 1.1) of the definition of a Γ-cut equation of rank L and size l.

□

Agreement 1 on $\mathcal{P}$. Fix an arbitrary integer l, $l \geq 5$. We may assume, choosing the constant a to satisfy the condition

$$a \geq (l + 2)N_{\Pi_\phi} + 3,$$

that all tuples in the set $\mathcal{P}$ are $[(l + 2)N_{\Pi_\phi} + 3]$-large. Denote $N = (l + 2)N_{\Pi_\phi}$.

Now we introduce one technical restriction on the set $\mathcal{P}$, its real meaning will be clarified later.

Agreement 2 on $\mathcal{P}$. Let r be an arbitrary fixed positive integer with $Kr \leq L$ and q be a fixed tuple of length Kr which is an initial segment of some tuple from $\mathcal{P}$. The choice of r and q will be clarified later. We may assume (suitably choosing the function h) that all tuples from $\mathcal{P}$ have q as their initial segment. Indeed, it suffices to define $h(i, 0) = 0$ and $h(i, j) = h(i+1, j)$ for all $i \in \mathbb{N}$ and $j = 1, \dots, Kr$.

Agreement 3 on $\mathcal{P}$. Let r be the number from Agreement 2. By Proposition 7.1 there exists a number a_0 such that for every infinite subset of $\mathcal{P}$ the corresponding set of solutions is a discriminating set. We may assume that $a > a_0$.

Transformation T^* of Γ-cut equations.

Now we describe a transformation T^* defined on Γ-cut equations and their solutions, namely, given a Γ-cut equation Π and its solution α (relative to the fixed map $\beta : F[X] \to F$ defined above) T^* transforms Π into a new Γ-cut equation $\Pi^* = T^*(\Pi)$ and α into a solution $\alpha^* = T^*(\alpha)$ of $T^*(\Pi)$ relative to β.

Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a Γ-cut equation in rank j and size l. The cut equation

$$T^*(\Pi) = (\mathcal{E}^*, f^*_{X^*}, f^*_{M^*})$$

is defined as follows.

Definition of the set $\mathcal{E}^*$.

For $\sigma \in \mathcal{E}$ we denote $W_\sigma = f_X(\sigma)$. Put

$$\mathcal{E}_{j,N} = \{\sigma \in \mathcal{E} \mid W_\sigma \text{ satisfies } 1.1)\}.$$

Then $\mathcal{E} = \mathcal{E}_{j,N} \cup \mathcal{E}_{<j,N}$ where $\mathcal{E}_{<j,N}$ is the complement of $\mathcal{E}_{j,N}$ in $\mathcal{E}$.

Now let $\sigma \in \mathcal{E}_{j,N}$. Write the word W_σ^β in its canonical A' decomposition:

$$W_\sigma^\beta = E_1 \circ A'^{q_1} \circ E_2 \circ \cdots \circ E_k \circ A'^{q_k} \circ E_{k+1} \tag{37}$$

where $|q_i| \geqslant 1$, $E_i \neq 1$ for $2 \leqslant i \leqslant k$.

Consider the partition

$$f_M(\sigma) = \mu_1 \dots \mu_n$$

of σ. By the condition 2) of the definition of Γ-cut equations for the solution $\beta : F[X] \to F$ there exists a solution $\alpha : F[M] \to F$ of the cut equation Π relative to β. Hence $W_\sigma^\beta = f_M(M^\alpha)$ and the element

$$f_M(M^\alpha) = \mu_1^\alpha \dots \mu_n^\alpha$$

is reduced as written. It follows that

$$W_\sigma^\beta = E_1 \circ {A'}^{q_1} \circ E_2 \circ \dots \circ E_k \circ {A'}^{q_k} \circ E_{k+1} = \mu_1^\alpha \circ \dots \circ \mu_n^\alpha. \tag{38}$$

We say that a variable μ_i is *long* if ${A'}^{\pm(l+2)}$ occurs in μ_i^α (i.e., μ_i^α contains a stable occurrence of ${A'}^l$), otherwise it is called *short*. Observe, that the definition of long (short) variables $\mu \in M$ does not depend on a choice of σ, it depends only on the given homomorphism α. The graphical equalities (38) (when σ runs over $\mathcal{E}_{j,N}$) allow one to effectively recognize long and short variables in M. Moreover, since for every $\sigma \in \mathcal{E}$ the length of the word $f_M(\sigma)$ is bounded by $length(\Pi) = N_\Pi$ and $N = (l+2)N_\Pi$, every word $f_M(\sigma)$ $(\sigma \in \mathcal{E}_j)$ contains long variables. Denote by M_{short}, M_{long} the sets of short and long variables in M. Thus, $M = M_{\text{short}} \cup M_{\text{long}}$ is a non-trivial partition of M.

Now we define the following property $P = P_{long,l}$ of occurrences of powers of A' in W_σ^β: a given stable occurrence ${A'}^q$ satisfies P if it occurs in μ^α for some long variable $\mu \in M_{long}$ and $q \geqslant l$. It is easy to see that P preserves correct overlappings. Consider the set of stable occurrences $\mathcal{O}_P$ which are maximal with respect to P. As we have mentioned already in Section 4, occurrences from $\mathcal{O}_P$ are pair-wise disjoint and this set is uniquely defined. Moreover, W_σ^β admits the unique A'-decomposition relative to the set $\mathcal{O}_P$:

$$W_\sigma^\beta = D_1 \circ (A')^{q_1} \circ D_2 \circ \dots \circ D_k \circ (A')^{q_k} \circ D_{k+1}, \tag{39}$$

where $D_i \neq 1$ for $i = 2, \dots, k$. See Figure 1.

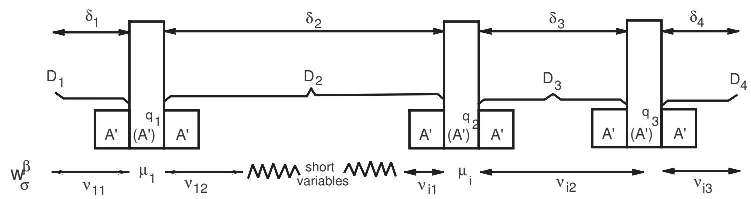

FIGURE 1. Decomposition (39)

Denote by $k(\sigma)$ the number of nontrivial elements among $D_1, \dots, D_{k+1}$.

According to Lemma 6.9 the A'-decomposition 39 gives rise to the unique associated A-decomposition of W_σ and hence the unique associated A^*-decomposition of W_σ.

Now with a given $\sigma \in \mathcal{E}_j$ we associate a finite set of new intervals E_σ (of the equation $T^*(\Pi)$):

$$E_\sigma = \{\delta_1, \dots, \delta_{k(\sigma)}\}$$

and put

$$\mathcal{E}^* = \mathcal{E}_{<j} \cup \bigcup_{\sigma \in \mathcal{E}_j} E_\sigma.$$

Definition of the set M^*

Let $\mu \in M_{long}$ and

$$\mu^\alpha = u_1 \circ (A')^{s_1} \circ u_2 \circ \cdots \circ u_t \circ (A')^{s_t} \circ u_{t+1} \tag{40}$$

be the canonical l-large A'-decomposition of μ^α. Notice that if μ occurs in $f_M(\sigma)$ (hence μ^α occurs in W_σ^β) then this decomposition (40) is precisely the A'-decomposition of μ^α induced on μ^α (as a subword of W_σ^β) from the A'-decomposition (39) of W_σ^β relative to $\mathcal{O}_P$.

Denote by $t(\mu)$ the number of non-trivial elements among $u_1, \ldots, u_{t+1}$ (clearly, $u_i \neq 1$ for $2 \leqslant i \leqslant t$).

We associate with each long variable μ a sequence of new variables (in the equation $T^*(\Pi)$) $S_\mu = \{\nu_1, \ldots, \nu_{t(\mu)}\}$. Observe, since the decomposition (40) of μ^α is unique, the set S_μ is well-defined (in particular, it does not depend on intervals σ).

It is convenient to define here two functions ν_{left} and ν_{right} on the set M_{long}: if $\mu \in M_{long}$ then

$$\nu_{\text{left}}(\mu) = \nu_1, \quad \nu_{\text{right}}(\mu) = \nu_{t(\mu)}.$$

Now we define a new set of variable M^* as follows:

$$M^* = M_{\text{short}} \cup \bigcup_{\mu \in M_{long}} S_\mu.$$

Definition of the labelling function $f^*_{X^*}$

Put $X^* = X$. We define the labelling function $f^*_{X^*} : \mathcal{E}^* \to F[X]$ as follows.

Let $\delta \in \mathcal{E}^*$. If $\delta \in \mathcal{E}_{<j}$, then put

$$f^*_{X^*}(\delta) = f_X(\delta).$$

Let now $\delta = \delta_i \in E_\sigma$ for some $\sigma \in M_{\text{long}}$. Then there are three cases to consider.

a) δ corresponds to the consecutive occurrences of powers ${A'}^{q_{j-1}}$ and ${A'}^{q_j}$ in the A'-decomposition (39) of W_σ^β relative to $\mathcal{O}_P$. Here $j = i$ or $j = i-1$ with respect to whether $D_1 = 1$ or $D_1 \neq 1$.

As we have mentioned before, according to Lemma 6.9 the A'-decomposition (39) gives rise to the unique associated A^*-decomposition of W_σ:

$$W_\sigma = D_1^* \circ_d (A^*)^{q_1^*} \circ_d D_2^* \circ \cdots \circ_d D_k^* \circ_d (A^*)^{q_k^*} \circ_d D_{k+1}^*.$$

Now put

$$f^*_X(\delta_i) = D_j^* \in F[X]$$

where $j = i$ if $D_1 = 1$ and $j = i-1$ if $D_1 \neq 1$. See Figure 2.

The other two cases are treated similarly to case a).

b) δ corresponds to the interval from the beginning of σ to the first A' power ${A'}^{q_1}$ in the decomposition (39) of W_σ^β. Put

$$f^*_X(\delta) = D_1^*.$$

c) δ corresponds to the interval from the last occurrence of a power ${A'}^{q_k}$ of A' in the decomposition (39) of W_σ^β to the end of the interval. Put

$$f^*_X(\delta) = D_{k+1}^*.$$

Definition of the function $f^*_{M^*}$.

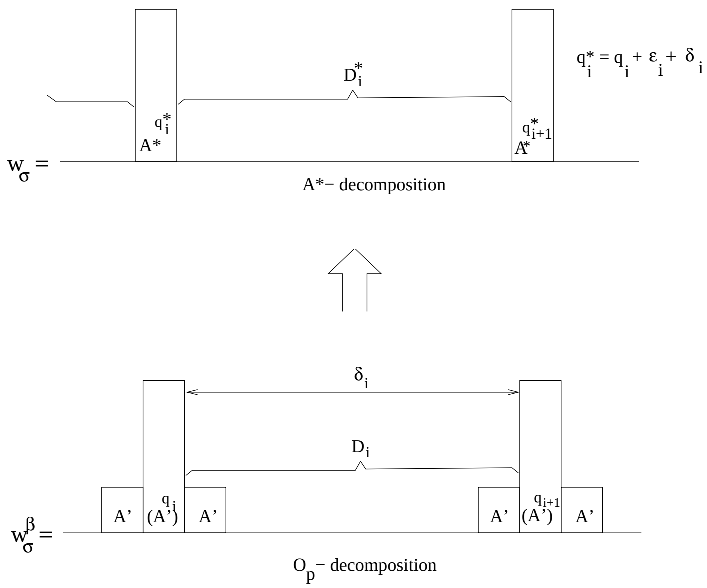

FIGURE 2. Defining $f^*_{X^*}$.

Now we define the function $f^* : \mathcal{E}^* \to F[M^*]$.

Let $\delta \in \mathcal{E}^*$. If $\delta \in \mathcal{E}_{<j}$, then put

$$f^*_{M^*}(\delta) = f_M(\delta)$$

(observe that all variables in $f_M(\delta)$ are short, hence they belong to M^*).

Let $\delta = \delta_i \in E_\sigma$ for some $\sigma \in M_{long}$. Again, there are three cases to consider.

a) δ corresponds to the consecutive occurrences of powers A'^{q_s} and $A'^{q_{s+1}}$ in the A'-decomposition (39) of W_σ^β relative to $\mathcal{O}_P$. Let the stable occurrence A'^{q_s} occur in μ_i^α for a long variable μ_i, and the stable occurrence $A'^{q_{s+1}}$ occur in μ_j^α for a long variable μ_j.

Observe that

$$D_s = right(\mu_i) \circ \mu_{i+1}^\alpha \circ \cdots \circ \mu_{j-1}^\alpha \circ left(\mu_j),$$

for some elements $right(\mu_i), left(\mu_j) \in F$.

Now put

$$f^*_{M^*}(\delta) = \nu_{i,\mathrm{right}}\mu_{i+1} \ldots \mu_{j-1}\nu_{j,left},$$

See Figure 3.

The other two cases are treated similarly to case a).

b) δ corresponds to the interval from the beginning of σ to the first A' power A'^{q_1} in the decomposition (39) of W_σ^β. Put

$$f^*_{M^*}(\delta) = \mu_1 \ldots \mu_{j-1}\nu_{j,left}.$$

c) δ corresponds to the interval from the last occurrence of a power A'^{q_k} of A' in the decomposition (39) of W_σ^β to the end of the interval.

The cut equation $T^*(\Pi) = (\mathcal{E}^*, f^*_X, f^*_{M^*})$ has been defined.

We define now a sequence

$$\Pi_L \xrightarrow{T^*} \Pi_{L-1} \xrightarrow{T^*} \ldots \xrightarrow{T^*} \Pi_1 \tag{41}$$

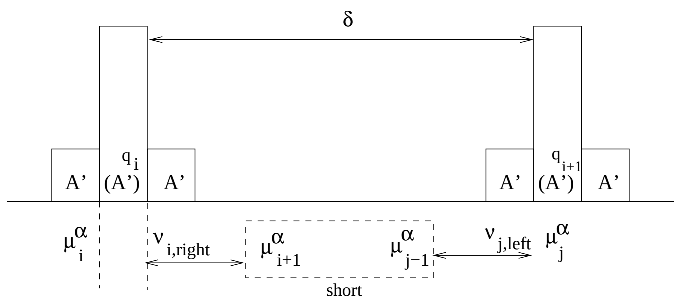

FIGURE 3. Defining $f^*_{M^*}$, case a)

of N-large Γ-cut equations, where $\Pi_L = \Pi_\phi$, and $\Pi_{i-1} = T^*(\Pi_i)$. In Claims 4 and 5 below we show that in this case if Π is a Γ-cut equation then $T^*(\Pi)$ is also a Γ-cut equation of the corresponding rank and size, so the sequence is well-defined. However, it is convenient to assume this as a fact now and introduce some notation and agreements before proving the claims.

CLAIM 1. *Let Π_j be a cut equation from the sequence (41). Then there exists an infinite subset $\mathcal{P}' \subseteq \mathcal{P}$ such that the cut equation $\Pi_{j-1} = T^*(\Pi_j)$ satisfies the following conditions:*

(1) *the words $f_{X^*}(\sigma) \in F[X]$, as parametric words in the parameters from p, are the same for every $p \in \mathcal{P}'$, i.e., they differ only in exponents corresponding to components of the tuples p.*
(2) *the words $f_{M^*}(\sigma)$ are the same for every $p \in \mathcal{P}'$.*

PROOF. The claim follows from the construction of $T^*(\Pi)$. □

Agreement 4 on the set $\mathcal{P}$: we assume (replacing P with a suitable infinite subset) that every tuple $p \in \mathcal{P}$ satisfies the conditions of Claim 1. Thus, every $\Pi = \Pi_i$ from the sequence (41) satisfies the conclusion of Claim 1 for $\mathcal{P}' = \mathcal{P}$.

CLAIM 2. *The homomorphism $\alpha^* : F[M^*] \to F$ defined as (in the notations above):*

$$\alpha^*(\mu) = \alpha(\mu) \quad (\mu \in M_{\text{short}}),$$
$$\alpha^*(\nu_{i,right}) = R^{-\beta}c^{-1}right(\mu_i) \quad (\nu_i \in S_\mu \ for \ \mu \in M_{long})$$
$$\alpha^*(\nu_{i,left}) = left(\mu_i)cR^\beta$$

is a solution of the cut equation $T^*(\Pi)$ with respect to $\beta : F[X] \to F$.

PROOF. Indeed, by Lemma 6.8

$$(D_s^*)^\beta = (R^{-\beta}c^{-1}) \circ_\theta D_s \circ_\theta (cR^\beta)$$

where $\theta << |A'|$. Therefore, $\mu^\alpha_{i+1} \dots \mu^\alpha_{j-1}$ occurs in D_s without cancellation. Therefore α^* is a required solution. □

Agreement 5 on the set $\mathcal{P}$: we assume (by choosing the function h properly, i.e., $h(i,j) > C(L, N+3)$, see Lemma) that every tuple $p \in \mathcal{P}$ satisfies the conditions of Lemma 6.8, so Claim 2 holds for every $p \in \mathcal{P}$. Thus, for every $\Pi = \Pi_i$ from the sequence (41) with a solution α (relative to β) the solution α^* of the equation $T^*(\Pi)$ is defined as in Claim 2.

CLAIM 3. *Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a Γ-cut equation in rank $j \geq 1$ from the sequence (41). Then for every variable $\mu \in M$ there exists a word $\mathcal{M}_\mu(M_{T(\Pi)}, X^{\phi_{j-1}}, F)$ such that the following equality holds in the group F*

$$\mu^\alpha = \mathcal{M}_\mu(M_{T(\Pi)}^{\alpha^*}, X^{\phi_{j-1}})^\beta.$$

Moreover, there exists an infinite subset $P' \subseteq P$ such that the words $\mathcal{M}_\mu(M_{T(\Pi)}, X)$ depend only on exponents $s_1, \ldots, s_t$ of the canonical l-large decomposition (40) of the words μ^α.

PROOF. The claim follows from the construction. Indeed, in constructing $T(\Pi)$ we cut out leading periods of the type $(A_j')^s$ from μ^α (see (40)). It follows that to get μ^α back from $M_{T(\Pi)}^{\alpha^*}$ one needs to put the exponents $(A_j')^s$ back. Notice, that

$$A_j = A(\gamma_j)^{\phi_{j-1}}$$

Therefore,

$$(A_j)^s = A(\gamma_j)^{\phi_{j-1}\beta}$$

Recall that A_j' is the cyclic reduced form of A_j^β, so

$$(A_j')^s = uA(\gamma_j)^{\phi_{j-1}\beta}v$$

for some constants $u, v \in C_\beta \subseteq F$. To see existence of the subset $P' \subseteq P$ observe that the length of the words $f_M(\sigma)$ does not depend on p, so there are only finitely many ways to cut out the leading periods $(A_j')^s$ from μ^α. This proves the claim. $\square$

Agreement 6 on the set $\mathcal{P}$: we assume (replacing P with a suitable infinite subset) that every tuple $p \in \mathcal{P}$ satisfies the conditions of Claim 3. Thus, for every $\Pi = \Pi_i$ from the sequence (41) with a solution α (relative to β) the solution α^* satisfies the conclusion of Claim 3.

DEFINITION 7.6. We define a new transformation T which is a modified version of T^*. Namely, T transforms cut equations and their solutions α precisely as the transformation T^*, but it also transforms the set of tuples $\mathcal{P}$ producing an infinite subset $\mathcal{P}^* \subseteq \mathcal{P}$ which satisfies the Agreements 1-6.

Now we define a sequence

$$\Pi_L \xrightarrow{T} \Pi_{L-1} \xrightarrow{T} \ldots \xrightarrow{T} \Pi_1 \tag{42}$$

of N-large Γ-cut equations, where $\Pi_L = \Pi_\phi$, and $\Pi_{i-1} = T(\Pi_i)$. From now on we fix the sequence (42) and refer to it as the *T-sequence.*

CLAIM 4. *The following statements are true:*

1) for every $i = 1, \ldots, L/K$ and every interval σ of the cut equation Π_{L-iK} from the T-sequence (42) there exists a word $w = w_\sigma \in \bar{\mathcal{W}}_{\Gamma,L}$ without N-large powers of elementary periods such that $f_X(\sigma) = w^{\phi_{L-iK}}$;

2) for every $j = 1, \ldots, L$ and every interval σ of the cut equation Π_{L-j} from the T-sequence (42) the label $f_X(\sigma)$ of σ belongs to $\bar{\mathcal{W}}_{\Gamma,L}$.

PROOF. We prove the claim by induction on i.

Let $i = 1$. For every $x \in X^{\pm 1}$ one can represent the element x^{ϕ_L} as a product of elements of the type $y^{\phi_{L-K}}, y \in X^{\pm 1}$ (in this event we say that the element x^{ϕ_L} is a word in the alphabet $X^{\phi_{L-K}}$). Indeed,

$$x^{\phi_L} = (x^{\phi_K})^{\phi_{L-K}} = w^{\phi_{L-K}},$$

where $w = x^{\phi_K}$ is a word in X.

Now consider the first K terms in the T-sequence:

$$\Pi_L \to \ldots \to \Pi_{L-K}.$$

We use induction on m to prove that for every interval $\sigma \in \Pi_{L-m} = (\mathcal{E}^{(L-m)}, f_X^{(L-m)}, f_M^{(L-m)})$ the label $f_X^{(L-m)}(\sigma)$ is of the form $u^{\phi_{L-K}}$ for some $u \in Sub(X^{\phi_K})$.

For $m = 1$ by Lemma 4.38 for $j = L, r = K$, there is a precise correspondence between stable A_L^*-decompositions of

$$x^{\phi_L} = w^{\phi_{L-K}} = D_1^{\phi_{L-K}} \circ_d A_L^{*q_1} \circ_d D_2^{\phi_{L-K}} \ldots D_k^{\phi_{L-K}} \circ_d A_L^{*q_k} \circ D_{k+1}^{\phi_{L-K}}$$

and stable A_K-decompositions of w

$$w = D_1 \circ A_K{}^{q_1} \circ D_2 \ldots D_k \circ A_K^{q_k} \circ D_{k+1}.$$

By construction, application of the transformation T to Π_L removes powers $A_L^{*q_s} = A_K^{q_s \phi_{L-K}}$ which are subwords of the word $w^{\phi_{L-K}}$ written in the alphabet $X^{\phi_{L-K}}$. By construction the words $D_s^{\phi_{L-K}}$ are the labels of the new intervals of the equation Π_{L-1}. Suppose by induction that for an interval σ of the cut equation Π_j (for $m = L - j$) $f_X^{(j)}(\sigma) = u^{\phi_{L-K}}$ for some $u \in Sub(X^{\pm\phi_K})$. Then either σ does not change under T or $f_X^{(j)}(\sigma)$ has a stable $(l+2)$-large $A_j{}^*$-decomposition in rank $j = r + (L-K)$ associated with long variables in $f_M^{(j)}(\sigma)$:

$$u^{\phi_{L-K}} = \bar{D}_1^{\phi_{L-K}} \circ_d A_j^{*q_1} \circ_d \bar{D}_2^{\phi_{L-K}} \ldots \bar{D}_k^{\phi_{L-K}} \circ_d A_j^{*q_k} \circ \bar{D}_{k+1}^{\phi_{L-K}},$$

and σ is an interval in Π_j. By Lemma 4.38, in this case there is a stable A_r-decomposition of u:

$$u = \bar{D}_1 \circ A_r^{q_1} \circ \bar{D}_2 \ldots \bar{D}_k \circ A_r^{q_k} \circ \bar{D}_{k+1}.$$

The application of the transformation T to Π_j removes powers $A_j^{*q_s} = A_r^{q_s \phi_{L-K}}$ (since $A_j{}^* = A_r^{\phi_{L-K}}$) which are subwords of the word $u^{\phi_{L-K}}$ written in the alphabet $X^{\phi_{L-K}}$. By construction the words $\bar{D}_s^{\phi_{L-K}}$ are the labels of the new intervals of the equation Π_{j-1}, so they have the required form. By induction the statement holds for $m = K$, so the label $f_X^{(L-K)}(\sigma)$ of an interval σ in Π_{L-K} is of the form $u^{\phi_{L-K}}$, for some $u \in Sub(X^{\pm\phi_K})$. Notice that $Sub(X^{\pm\phi_K}) \subseteq \mathcal{W}_{\Gamma,L}$ which proves statement 1) of the Claim for $i = 1$ and proves the statement 2) for all $j = 1, \ldots, K$.

Suppose, by induction, that labels of intervals in the cut equation Π_{L-Ki} have form $w^{\phi_{L-Ki}}$, $w \in \bar{\mathcal{W}}_{\Gamma,L}$. We can rewrite each label in the form $v^{\phi_{L-K(i+1)}}$, where $v = w^{\phi_K} \in \bar{\mathcal{W}}_{\Gamma,L}$. In the T-sequence

$$\Pi_{L-Ki} \to \ldots \to \Pi_{L-K(i+1)}$$

each application of the transformation T removes subwords in the alphabet $X^{\phi_{L-K(i+1)}}$. The argument above shows that the labels of the new intervals in all cut equations $\Pi_{L-Ki-1)}, \ldots, \Pi_{L-K(i+1)}$ are of the form $v^{\phi_{L-K(i+1)}}$, where $v \in \bar{\mathcal{W}}_{\Gamma,L}$. Following the proof it is easy to see that the word v does not contain N-large powers of $e^{\phi_{L-K(i+1)}}$ for an elementary period e.

$\square$

Claim 5. *Let $l \geq 3$, $p_{j-1} \geq (l+2)N_\Pi + 3$. The cut equation $T(\Pi)$ is a Γ-cut equation in rank $\leq j-1$ of size l.*

PROOF. The claim follows from the construction of $T(\Pi)$. More precisely, we show first that $T(\Pi)$ has a solution relative to β. It has been shown in Claim 1 that $T^*(\Pi)$ has a solution α^* relative to β. This proves condition 2) in the definition of the Γ-cut equation.

Observe also, that to show 1) it suffices to show that 1.1) in rank j does not hold for $T^*(\Pi)$. It is not hard to see that it suffices to prove the required inequalities for A'-decompositions (see Lemma 6.9).

Let $\delta \in \mathcal{E}^*$. By the construction $(A')^{l+2}$ does not occur in μ^α for any $\mu \in M^*$. Therefore the maximal power of A' that can occur in $f^*_{M^*}(\delta)^\alpha$ is bounded from above by $(l+1)|f^*_{M^*}(\delta)|$ which is less then $(l+1)length(T^*(\Pi))$, as required. Let t be the rank of $T(\Pi)$, $t \leq j-1$. It follows from the construction that if conditions 1.1) and 1.3) for rank t are not satisfied for an interval in $T(\Pi)$, then condition 1.2) is satisfied. □

DEFINITION 7.7. Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a cut equation. For a positive integer n by $k_n(\Pi)$ we denote the number of intervals $\sigma \in \mathcal{E}$ such that $|f_M(\sigma)| = n$. The following finite sequence of integers

$$Comp(\Pi) = (k_2(\Pi), k_3(\Pi), \ldots, k_{length(\Pi)}(\Pi))$$

is called the *complexity* of Π.

We well-order complexities of cut equations in the (right) shortlex order: if Π and Π' are two cut equations then $Comp(\Pi) < Comp(\Pi')$ if and only if $length(\Pi) < length(\Pi')$ or $length(\Pi) = length(\Pi')$ and there exists $1 \leqslant i \leqslant length(\Pi)$ such that $k_j(\Pi) = k_j(\Pi')$ for all $j > i$ but $k_i(\Pi) < k_i(\Pi')$.

Observe that intervals $\sigma \in \mathcal{E}$ with $|f_M(\sigma)| = 1$ have no input into the complexity of a cut equation $\Pi = (\mathcal{E}, f_X, f_M)$. In particular, equations with $|f_M(\sigma)| = 1$ for every $\sigma \in \mathcal{E}$ have the minimal possible complexity among equations of a given length. We will write $Comp(\Pi) = \mathbf{0}$ in the case when $k_i(\Pi) = 0$ for every $i = 2, \ldots, length(\Pi)$.

CLAIM 6. *Let $\Pi = (\mathcal{E}, f_X, f_M)$. Then the following holds:*
(1) *$length(T(\Pi)) \leqslant length(\Pi)$;*
(2) *$Comp(T(\Pi)) \leqslant Comp(\Pi)$.*

PROOF. By straightforward verification. Indeed, if $\sigma \in \mathcal{E}_{<j}$ then $f_M(\sigma) = f^*_{M^*}(\sigma)$. If $\sigma \in \mathcal{E}_j$ and $\delta_i \in E_\sigma$ then

$$f^*_{M^*}(\delta_i) = \mu^*_{i_1}\mu_{i_1+1}\ldots\mu^*_{i_1+r(i)},$$

where $\mu_{i_1}\mu_{i_1+1}\ldots\mu_{i_1+r(i)}$ is a subword of $\mu_1\ldots\mu_n$ and hence $|f^*_{M^*}(\delta_i)| \leqslant |f_M(\sigma)|$, as required. □

We need a few definitions related to the sequence (42). Denote by M_j the set of variables in the equation Π_j. Variables from Π_L are called *initial* variables. A variable μ from M_j is called *essential* if it occurs in some $f_{M_j}(\sigma)$ with $|f_{M_j}(\sigma)| \geqslant 2$, such occurrence of μ is called *essential*. By $n_{\mu,j}$ we denote the total number of all essential occurrences of μ in Π_j. Then

$$S(\Pi_j) = \sum_{i=2}^{N_{\Pi_j}} ik_i(\Pi_j) = \sum_{\mu \in M_j} n_{\mu,j}$$

is the total number of all essential occurrences of variables from M_j in Π_j.

Claim 7. *If* $1 \leqslant j \leqslant L$ *then* $S(\Pi_j) \leqslant 2S(\Pi_L)$.

Proof. Recall, that every variable μ in M_j either belongs to M_{j+1} or it is replaced in M_{j+1} by the set S_μ of new variables (see definition of the function $f^*_{M^*}$ above). We refer to variables from S_μ as to *children* of μ. A given occurrence of μ in some $f_{M_{j+1}}(\sigma)$, $\sigma \in \mathcal{E}_{j+1}$, is called a *side occurrence* if it is either the first variable or the last variable (or both) in $f_{M_{j+1}}(\sigma)$. Now we formulate several properties of variables from the sequence (42) which come directly from the construction. Let $\mu \in M_j$. Then the following conditions hold:

(1) every child of μ occurs only as a side variable in Π_{j+1};
(2) every side variable μ has at most one essential child, say μ^*. Moreover, in this event $n_{\mu^*,j+1} \leqslant n_{\mu,j}$;
(3) every initial variable μ has at most two essential children, say μ_{left} and μ_{right}. Moreover, in this case $n_{\mu_{\text{left}},j+1} + n_{\mu_{\text{right}},j+1} \leqslant 2n_\mu$.

Now the claim follows from the properties listed above. Indeed, every initial variable from Π_j doubles, at most, the number of essential occurrences of its children in the next equation Π_{j+1}, but all other variables (not the initial ones) do not increase this number. □

Denote by *width*(Π) the *width* of Π which is defined as

$$width(\Pi) = \max_i k_i(\Pi).$$

Claim 8. *For every* $1 \leqslant j \leqslant L$ *width*$(\Pi_j) \leqslant 2S(\Pi_L)$

Proof. It follows directly from Claim 7. □

Denote by $\kappa(\Pi)$ the number of all $(length(\Pi)-1)$-tuples of non-negative integers which are bounded by $2S(\Pi_L)$.

Claim 9. $Comp(\Pi_L) = Comp(\Pi_{\mathcal{L}})$.

Proof. The complexity $Comp(\Pi_L)$ depends only on the function f_M in Π_L. Recall that $\Pi_L = \Pi_\phi$ is obtained from the cut equation $\Pi_{\mathcal{L}}$ by changing only the labelling function f_X, so $\Pi_{\mathcal{L}}$ and Π_L have the same functions f_M, hence the same complexities. □

We say that a T-sequence has $3K$*-stabilization* at $K(r+2)$, where $2 \leqslant r \leqslant L/K$, if

$$Comp(\Pi_{K(r+2)}) = \ldots = Comp(\Pi_{K(r-1)}).$$

In this event we denote

$$K_0 = K(r+2), \quad K_1 = K(r+1), \quad K_2 = Kr, \quad K_3 = K(r-1).$$

For the cut equation Π_{K_1} by $M_{\text{veryshort}}$ we denote the subset of variables from $M(\Pi_{K_1})$ which occur unchanged in Π_{K_2} and are short in Π_{K_2}.

Claim 10. *For a given* Γ*-cut equation* Π *and a positive integer* $r_0 \geqslant 2$ *if* $L \geqslant Kr_0 + \kappa(\Pi)4K$ *then for some* $r \geqslant r_0$ *either the sequence* (42) *has* $3K$*-stabilization at* $K(r+2)$ *or* $Comp(\Pi_{K(r+1)}) = 0$.

Proof. Indeed, the claim follows by the "pigeon hole" principle from Claims 6 and 8 and the fact that there are not more than $\kappa(\Pi)$ distinct complexities which are less or equal to $Comp(\Pi)$. □

Now we define a special set of solutions of the equation $S(X) = 1$. Let $L = 4K + \kappa(\Pi)4K$, p be a fixed N-large tuple from $\mathbb{N}^{L-4K}$, q be an arbitrary fixed N-large tuple from $\mathbb{N}^{2K}$, and p^* be an arbitrary N-large tuple from $\mathbb{N}^{2K}$. In fact, we need N-largeness of p^* and q only to formally satisfy the conditions of the claims above. Put

$$\mathcal{B}_{p,q,\beta} = \left\{\phi_{L-4K,p}\phi_{2K,p^*}\phi_{2K,q}\beta \mid p^* \in \mathbb{N}^{2K}, pp^*q \in \mathcal{P}\right\}.$$

It follows from Theorem 5.3 that $\mathcal{B}_{p,q,\beta}$ is a discriminating family of solutions of $S(X) = 1$.

Denote $\beta_q = \phi_{2K,q} \circ \beta$. Then β_q is a solution of $S(X) = 1$ in general position and

$$\mathcal{B}_{q,\beta} = \{\phi_{2K,p^*}\beta_q \mid p^* \in \mathbb{N}^{2K}\}$$

is also a discriminating family by Theorem 5.3.

Let

$$\mathcal{B} = \{\psi_{K_1} = \phi_{K(r-2),p'}\phi_{2K,p^*}\phi_{2K,q}\beta \mid p^* \in \mathbb{N}^{2K}\},$$

where p' is a beginning of p.

Proposition 7.8. *Let $L = 2K + \kappa(\Pi)4K$ and $\phi_L \in \mathcal{B}_{p,q,\beta}$. Suppose the T-sequence of cut equations (42) has $3K$-stabilization at $K(r+2), r \geqslant 2$. Then the set of variables M of the cut equation $\Pi_{K(r+1)}$ can be partitioned into three disjoint subsets*

$$M = M_{\text{veryshort}} \cup M_{\text{free}} \cup M_{\text{useless}}$$

for which the following holds:

(1) *there exists a finite system of equations $\Delta(M_{\text{veryshort}}) = 1$ over F which has a solution in F;*
(2) *for every $\mu \in M_{\text{useless}}$ there exists a word $V_\mu \in F[X \cup M_{\text{free}} \cup M_{\text{veryshort}}]$ which does not depend on tuples p^* and q;*
(3) *for every solution $\delta \in \mathcal{B}$, for every map $\alpha_{\text{free}} : M_{\text{free}} \to F$, and every solution $\alpha_s : F[M_{\text{veryshort}}] \to F$ of the system $\Delta(M_{\text{veryshort}}) = 1$ the map $\alpha : F[M] \to F$ defined by*

$$\mu^\alpha = \begin{cases} \mu^{\alpha_{\text{free}}} & \text{if } \mu \in M_{\text{free}}; \\ \mu^{\alpha_s} & \text{if } \mu \in M_{\text{veryshort}}; \\ V_\mu(X^\delta, M_{\text{free}}^{\alpha_{\text{free}}}, M_{\text{veryshort}}^{\alpha_s}) & \text{if } \mu \in M_{\text{useless}}. \end{cases}$$

is a group solution of $\Pi_{K(r+1)}$ with respect to β.

Proof. Below we describe (in a series of claims 11-22) some properties of partitions of intervals of cut equations from the sequence (42):

$$\Pi_{K_1} \xrightarrow{T} \Pi_{K_1-1} \xrightarrow{T} \ldots \xrightarrow{T} \Pi_{K_2}.$$

Fix an arbitrary integer s such that $K_1 \geqslant s \geqslant K_2$.

Claim 11. *Let $f_M(\sigma) = \mu_1 \cdots \mu_k$ be a partition of an interval σ of rank s in Π_s. Then:*

(1) *the variables $\mu_2, \ldots, \mu_{k-1}$ are very short;*
(2) *either μ_1 or μ_k, or both, are long variables.*

PROOF. Indeed, if any of the variables $\mu_2, \ldots, \mu_{k-1}$ is long then the interval σ of Π_s is replaced in $T(\Pi_s)$ by a set of intervals E_σ such that $|f_M(\delta)| < |f_M(\sigma)|$ for every $\delta \in E_\sigma$. This implies that complexity of $T(\Pi_s)$ is smaller than of Π_s - contradiction. On the other hand, since σ is a partition of rank s some variables must be long - hence the result. $\square$

Let $f_M(\sigma) = \mu_1 \ldots \mu_k$ be a partition of an interval σ of rank s in Π_s. Then the variables μ_1 and μ_k are called *side variables.*

CLAIM 12. *Let $f_M(\sigma) = \mu_1 \ldots \mu_k$ be a partition of an interval σ of rank s in Π_s. Then this partition will induce a partition of the form $\mu_1'\mu_2 \ldots \mu_{k-1}\mu_k'$ of some interval in rank $s-1$ in Π_{s-1} such that if μ_1 is short in rank s then $\mu_1' = \mu_1$, if μ_1 is long in Π_s then μ_1' is a new variable which does not appear in the previous ranks. Similar conditions hold for μ_k.*

PROOF. Indeed, this follows from the construction of the transformation T. $\square$

CLAIM 13. *Let σ_1 and σ_2 be two intervals of ranks s in Π_s such that $f_X(\sigma_1) = f_X(\sigma_2)$ and*

$$f_M(\sigma_1) = \mu_1\nu_2 \ldots \nu_k, \quad f_M(\sigma_2) = \mu_1\lambda_2 \ldots \lambda_l.$$

Then for any solution α of Π_s one has

$$\nu_k^\alpha = \nu_{k-1}^{-\alpha} \ldots \nu_2^{-\alpha}\lambda_2^{-\alpha} \ldots \lambda_{l-1}^{-\alpha}\lambda_l^{-\alpha}$$

i.e, ν_k^α can be expressed via λ_l^α and a product of images of short variables.

CLAIM 14. *Let $f_M(\sigma) = \mu_1 \ldots \mu_k$ be a partition of an interval σ of rank s in Π_s. Then for any $u \in X \cup E(m,n)$ the word $\mu_2^\alpha \ldots \mu_{k-1}^\alpha$ does not contain a subword of the type $c_1(M_u^{\phi_{K_1}})^\beta c_2$, where $c_1, c_2 \in C_\beta$, and $M_u^{\phi_{K_1}}$ is the middle of u with respect to ϕ_{K_1}.*

PROOF. By Corollary 4.22 every word $M_u^{\phi_{K_1}}$ contains a big power (greater than $(l+2)N_{\Pi_s}$) of a period in rank strictly greater than K_2. Therefore, if $(M_u^{\phi_{K_1}})^\beta$ occurs in the word $\mu_2^\alpha \ldots \mu_{k-1}^\alpha$ then some of the variables $\mu_2, \ldots, \mu_{k-1}$ are not short in some rank greater than K_2 - contradiction. $\square$

CLAIM 15. *Let σ be an interval in Π_{K_1} and $\phi_{K_1} = \phi_{K_1,p}$. Then $f_X(\sigma) = W_\sigma$ written in the form*

$$W_\sigma = w^{\phi_{K_1}},$$

and the following holds:

(1) *the word w can be uniquely written as $w = v_1 \ldots v_e$, where $v_1, \ldots v_e \in X^{\pm 1} \cup E(m,n)^{\pm 1}$, and $v_iv_{i+1} \notin E(m,n)^{\pm 1}$.*
(2) *w is either a subword of a word from the list in Lemma 4.16 or there exists i such that $v_1 \cdots v_i$, $v_{i+1} \cdots v_e$ are subwords of words from the list in Lemma 4.26. In addition, $(v_iv_{i+1})^{\phi_K} = v_i^{\phi_K} \circ v_{i+1}^{\phi_K}$.*
(3) *if w is a subword of a word from the list in Lemma 4.16, then at most for two indices i, j elements v_i, v_j belong to $E(m,n)^{\pm 1}$, and, in this case $j = i+1$.*

PROOF. The fact that W_σ can be written in such a form follows from Claim 4 for $r = K$. Indeed, by Claim 4, $W_\sigma = w^{\phi_{K_1}}$, where $w \in \mathcal{W}_{\Gamma,L}$, therefore it is either a subword of a word from the list in Lemma 4.16 or contains a subword from the set *Exc* from Lemma 4.26. It can contain only one such subword, because two

such subwords of a word from $X^{\pm\phi_L}$ are separated by big (unbounded) powers of elementary periods. The uniqueness of w in the first statement follows from the fact that $\phi_{K_1,p'}$ is an automorphism. Obviously, w does not depend on p. Property (3) follows from the comparison of the set $E(m,n)$ with the list from Lemma 4.16. $\square$

We say that the decomposition $w = v_1 \cdots v_e$, above is the *canonical decomposition* of w and $(v_1 \ldots v_s)^{\phi_{K_1}}$ is a canonical decomposition of $w^{\phi_{K_1}}$.

CLAIM 16. *Let $\Pi_{K_1} = (\mathcal{E}, f_X, f_M)$ and $\mu \in M$ be a long variable (in rank K_1) such that $f_M(\delta) \neq \mu$ for any $\delta \in \mathcal{E}$. If μ occurs as the left variable in $f_M(\sigma)$ for some $\delta \in \mathcal{E}$ then it does not occur as the right variable in $f_M(\delta)$ for any $\delta \in \mathcal{E}$ (however, μ^{-1} can occur as the right variable). Similarly, If μ occurs as the right variable in $f_M(\sigma)$ then it does not occur as the right variable in any $f_M(\delta)$.*

PROOF. Notice, that in this case if μ_1 is not a single variable, it cannot be a right side variable of $f_M(\bar{\sigma})$ for some interval $\bar{\sigma}$. Indeed, suppose $W_{\bar{\sigma}}$ ends with μ_1. If $v_{left} \neq z_i, y_n^{-1}$, W_σ begins with a big power of some period $A_j^{*\beta}$, $j > K_2$, therefore μ_1 begins with this big power, and the complexity of $\bar{\sigma}$ would decrease when we apply T to the cut equation in rank j. If $v_{left} = z_i$, μ_1 cannot be the right side variable, because c_i^N can occur only in the beginning of labels of intervals. If $v_{left} = y_n^{-1}$, then $W_{\bar{\sigma}} = \cdots x_n^{-1} \circ y_n^{-1}$, and the complexity would also decrease when T is applied in rank $K_2 + m + 4n - 4$. $\square$

Our next goal is to transform further the cut equation Π_{K_1} to the form where all intervals are labelled by elements $x^{\phi_{K_1}}$, $x \in (X \cup E(m,n))^{\pm 1}$. To this end we introduce several new transformations of Γ-cut equations.

Let $\Pi = (\mathcal{E}, f_X, f_M)$ be a Γ-cut equation in rank K_1 and size l with a solution $\alpha : F[M] \to F$ relative to $\beta : F[X] \to F$. Let $\sigma \in \mathcal{E}$ and

$$W_\sigma = (v_1 \cdots v_e)^{\phi_{K_1}}, \quad e \geq 2,$$

be the canonical decomposition of W_σ. For $i, 1 \leq i < e$, put

$$v_{\sigma,i,left} = v_1 \cdots v_i, \; v_{\sigma,i,right} = v_{i+1} \cdots v_e.$$

Let, as usual,

$$f_M(\sigma) = \mu_1 \cdots \mu_k.$$

We start with a **transformation** $T_{1,left}$. For $\sigma \in \mathcal{E}$ and $1 \leq i < e$ denote by θ the boundary between $v_{\sigma,i,left}^{\phi_{K_1}\beta}$ and $v_{\sigma,i,right}^{\phi_{K_1}\beta}$ in the reduced form of the product $v_{\sigma,i,left}^{\phi_{K_1}\beta} v_{\sigma,i,right}^{\phi_{K_1}\beta}$. Suppose now that there exist σ and i such that the following two conditions hold:

C1) μ_1^α almost contains the beginning of the word $v_{\sigma,i,left}^{\phi_{K_1}\beta}$ till the boundary θ (up to a very short end of it), i.e., there are elements $u_1, u_2, u_3, u_4 \in F$ such that $v_{\sigma,i,left}^{\phi_{K_1}\beta} = u_1 \circ u_2 \circ u_3$, $v_{i+1}^{\phi_{K_1}\beta} = u_3^{-1} \circ u_4$, $u_1 u_2 u_4 = u_1 \circ u_2 \circ u_4$, and μ_1^α begins with u_1, and u_2 is very short (does not contain $A_{K_2}^{\pm l}$) or trivial.

C2) the boundary θ does not lie inside μ_1^α.

In this event the transformation $T_{1,left}$ is applicable to Π as described below. We consider three cases with respect to the location of θ on $f_M(\sigma)$.

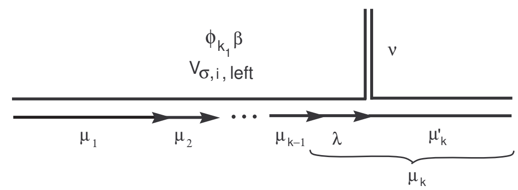

FIGURE 4. T2, Case 1)

Case 1) θ is inside μ_k^α (see Fig. 4). In this case we perform the following:

a) Replace the interval σ by two new intervals σ_1, σ_2 with the labels $v_{\sigma,i,left}^{\phi_{K_1}}$, $v_{\sigma,i,right}^{\phi_{K_1}}$;

b) Put $f_M(\sigma_1) = \mu_1 \dots \mu_{k-1}\lambda\nu$, $f_M(\sigma_2) = \nu^{-1}\mu_k'$, where λ is a new very short variable, ν is a new variable.

c) Replace everywhere μ_k by $\lambda\mu_k'$. This finishes the description of the cut equation $T_{1,left}(\Pi)$.

d) Define a solution α^* (with respect to β) of $T_{1,left}(\Pi)$ in the natural way. Namely, $\alpha^*(\mu) = \alpha(\mu)$ for all variables μ which came unchanged from Π. The values $\lambda^{\alpha^*}, {\mu'}_k^{\alpha^*}$, ν^{α^*} are defined in the natural way, that is ${\mu'}_k^{\alpha^*}$ is the whole end part of μ_k^α after the boundary θ, $(\nu^{-1}\mu_k')^{\alpha^*} = v_{\sigma,i,\mathrm{right}}^{\phi_{K_1}\beta}$, $\lambda^{\alpha^*} = \mu_k^\alpha({\mu'}_k^\alpha)^{-1}$.

Case 2) θ is on the boundary between μ_j^α and μ_{j+1}^α for some j. In this case we perform the following:

a) We split the interval σ into two new intervals σ_1 and σ_2 with labels $v_{\sigma,i,left}^{\phi_{K_1}}$ and $v_{\sigma,i,right}^{\phi_{K_1}}$.

b) We introduce a new variable λ and put $f_M(\sigma_1) = \mu_1 \dots \mu_j\lambda$, $f_M(\sigma_2) = \lambda^{-1}\mu_{j+1}\dots\mu_k$.

c) Define λ^{α^*} naturally.

Case 3) The boundary θ is contained inside μ_i^α for some $i (2 \leq i \leq r-1)$. In this case we do the following:

a) We split the interval σ into two intervals σ_1 and σ_2 with labels $v_{left}^{\phi_{K_1}}$ and $v_{\sigma,i,right}^{\phi_{K_1}}$, respectively.

b) Then we introduce three new variables μ_j', μ_j'', λ, where μ_j', μ_j'' are "very short", and add equation $\mu_j = \mu_j'\mu_j''$ to the system $\Delta_{\mathrm{veryshort}}$.

c) We define $f_M(\sigma_1) = \mu_1 \cdots \mu_j'\lambda$, $f_M(\sigma_2) = \lambda^{-1}\mu_j''\mu_{i+1}\cdots\mu_k$.

d) Define values of α^* on the new variables naturally. Namely, put λ^{α^*} to be equal to the terminal segment of $v_{left}^{\phi_{K_1}\beta}$ that cancels in the product $v_{left}^{\phi_{K_1}\beta}v_{\sigma,i,\mathrm{right}}^{\phi_{K_1}\beta}$. Now the values $\mu_j'^{\alpha^*}$ and $\mu_j''^{\alpha^*}$ are defined to satisfy the equalities

$$f_X(\sigma_1)^\beta = f_M(\sigma_1)^{\alpha^*}, f_X(\sigma_2)^\beta = f_M(\sigma_2)^{\alpha^*}.$$

We described the transformation $T_{1,left}$. The transformation $T_{1,right}$ is defined similarly. We denote both of them by T_1.

Now we describe a **transformation** $T_{2,left}$.

Suppose again that a cut equation Π satisfies C1). Assume in addition that for these σ and i the following condition holds:

C3) the boundary θ lies inside μ_1^{α}.

Assume also that one of the following three conditions holds:

C4) there are no intervals $\delta \neq \sigma$ in Π such that $f_M(\delta)$ begins with μ_1 or ends on μ_1^{-1};

C5) $v_{\sigma,i,left} \neq x_n$ (i.e., either $i > 1$ or $i = 1$ but $v_1 \neq x_n$) and for every $\delta \in \mathcal{E}$ in Π if $f_M(\delta)$ begins with μ_1 (or ends on μ_1^{-1}) then the canonical decomposition of $f_X(\delta)$ begins with $v_{\sigma,i,left}^{\phi_{K_1}}$ (ends with $v_{\sigma,i,left}^{-\phi_{K_1}}$);

C6) $v_{\sigma,i,left} = x_n$ ($i = 1$ and $v_1 = x_n$) and for every $\delta \in \mathcal{E}$ if $f_M(\delta)$ begins with μ_1 (ends with μ_i^{-1}) then the canonical decomposition of $f_X(\delta)$ begins with $x_n^{\phi_{K_1}}$ or with $y_n^{\phi_{K_1}}$ (ends with $x_n^{-\phi_{K_1}}$ or $y_n^{-\phi_{K_1}}$).

In this event the transformation $T_{2,left}$ is applicable to Π as described below.

Case C4) Suppose the condition C4) holds. In this case we do the following.

a) Replace σ by two new intervals σ_1, σ_2 with the labels $v_{\sigma,i,left}^{\phi_{K_1}}$, $v_{\sigma,i,right}^{\phi_{K_1}}$;

b) Replace μ_1 with two new variables μ_1', μ_1'' and put $f_M(\sigma_1) = \mu_1'$, $f_M(\sigma_2) = \mu_1''\mu_2 \dots \mu_k$.

c) Define $(\mu_1')^{\alpha^*}$ and $(\mu_1'')^{\alpha^*}$ such that $f_M(\sigma_1)^{\alpha^*} = v_{\sigma,i,left}^{\phi_{K_1}\beta}$ and $f_M(\sigma_2)^{\alpha^*} = v_{\sigma,i,right}^{\phi_{K_1}\beta}$.

Case C5) Suppose $v_{left} \neq x_n$. Then do the following.

a) Transform σ as described in C4).

b) If for some interval $\delta \neq \sigma$ the word $f_M(\delta)$ begins with μ_1 then replace μ_1 in $f_M(\delta)$ by the variable μ_1'' and replace $f_X(\delta)$ by $v_{\sigma,i,left}^{-\phi_{K_1}} f_X(\delta)$. Similarly transform intervals δ that end with μ_1^{-1}.

Case C6) Suppose $v_{left} = x_n$. Then do the following.

a) Transform σ as described in C4).

b) If for some δ the word $f_M(\delta)$ begins with μ_1 and $f_X(\delta)$ does not begin with y_n then transform δ as described in Case C5).

c) Leave all other intervals unchanged.

We described the transformation $T_{2,left}$. The transformation $T_{2,right}$ is defined similarly. We denote both of them by T_2.

Suppose now that $\Pi = \Pi_{K_1}$. Observe that the transformations T_1 and T_2 preserve the properties described in Claims 6–9 above. Moreover, for the homomorphism $\beta : F[X] \to F$ we have constructed a solution $\alpha^* : F[M] \to F$ of $T_n(\Pi_{K_1})$ ($n = 2, 3$) such that the initial solution α can be reconstructed from α^* and the equations Π and $T_n(\Pi)$. Notice also that the length of the elements $W_{\sigma'}$ corresponding to new intervals σ are shorter than the length of the words W_σ of the original intervals σ from which σ' were obtained. Notice also that the transformations T_1, T_2 preserves the property of intervals formulated in the Claim 11.

CLAIM 17. *Let Π be a cut equation which satisfies the conclusion of the Claim 11. Suppose σ is an interval in Π such that W_σ satisfies the conclusion of Claim 15. If*

for some i

$$(v_1 \dots v_e)^{\phi_K} = (v_1 \dots v_i)^{\phi_K} \circ (v_{i+1} \dots v_e)^{\phi_K}$$

then either T_1 or T_2 is applicable to given σ and i.

PROOF. By Corollary 4.36 the automorphism ϕ_{K_1} satisfies the Nielsen property with respect to $\bar{\mathcal{W}}_\Gamma$ with exceptions $E(m,n)$. By Corollary 12, equality

$$(v_1 \dots v_e)^{\phi_K} = (v_1 \dots v_i)^{\phi_K} \circ (v_{i+1} \dots v_e)^{\phi_K}$$

implies that the element that is cancelled between $(v_1 \dots v_i)^{\phi_K \beta}$ and $(v_{i+1} \dots v_e)^{\phi_K \beta}$ is short in rank K_2. Therefore either μ_1^α almost contains $(v_1 \dots v_i)^{\phi_K \beta}$ or μ_k^α almost contains $(v_{i+1} \dots v_e)^{\phi_K \beta}$. Suppose μ_1^α almost contains $(v_1 \dots v_i)^{\phi_K \beta}$. Either we can apply $T_{1,left}$, or the boundary θ belongs to μ_1^α. One can verify using formulas from Lemmas 4.6-4.9 and 4.21 directly that in this case one of the conditions $C4) - C6)$ is satisfied, and, therefore $T_{2,left}$ can be applied. $\square$

LEMMA 7.9. *Given a cut equation Π_{K_1} one can effectively find a finite sequence of transformations $Q_1, \dots, Q_s$ where $Q_i \in \{T_1, T_2\}$ such that for every interval σ of the cut equation $\Pi'_{K_1} = Q_s \dots Q_1(\Pi_{K_1})$ the label $f_X(\sigma)$ is of the form $u^{\phi_{K_1}}$, where $u \in X^{\pm 1} \cup E(m,n)$.*

Moreover, there exists an infinite subset P' of the solution set P of Π_{K_1} such that this sequence is the same for any solution in P'.

PROOF. Let σ be an interval of the equation Π_{K_1}. By Claim 15 the word W_σ can be uniquely written in the canonical decomposition form

$$W_\sigma = w^{\phi_{K_1}} = (v_1 \dots v_e)^{\phi_{K_1}},$$

so that the conditions 1), 2), 3) of Claim 15 are satisfied.

It follows from the construction of Π_{K_1} that either w is a subword of a word between two elementary squares $x \neq c_i$ or begins and (or) ends with some power ≥ 2 of an elementary period. If u is an elementary period, $u^{2\phi_K} = u^{\phi_K} \circ u^{\phi_K}$, except $u = x_n$, when the middle is exhibited in the proof of Lemma 4.21. Therefore, by Claim 17, we can apply T_1 and T_2 and cut σ into subintervals σ_i such that for any i $f_X(\sigma_i)$ does not contain powers ≥ 2 of elementary periods. All possible values of u^{ϕ_K} for $u \in E(m,n)^{\pm 1}$ are shown in the proof of Lemma 4.21. Applying T_1 and T_2 as in Claim 17 we can split intervals (and their labels) into parts with labels of the form $x^{\phi_{K_1}},\ x \in (X \cup E(m,n))$, except for the following cases:

1. $w = uv$, where u is $x_i^2, i < n,\ v \in E_{m,n}$, and v has at least three letters,
2. $w = x_{n-2}^2 y_{n-2} x_{n-1}^{-1} x_n x_{n-1} y_{n-2}^{-1} x_{n-2}^2$,
3. $w = x_{n-1}^2 y_{n-1} x_n^{-1} x_{n-1} y_{n-2}^{-1} x_{n-2}^{-2}$,
4. $y_{r-1} x_r^{-1} y_r^{-1},\ r < n$,
5. $w = uv$, where $u = (c_1^{z_1} c_2^{z_2})^2,\ v \in E(m,n)$, and v is one of the following: $v = \prod_{t=1}^m c_t^{z_t} x_1^{\pm 1}$, $v = \prod_{t=1}^m c_t^{z_t} x_1^{\pm 1} \prod_{t=m}^1 c_t^{-z_t}$, $v = \prod_{t=1}^m c_t^{z_t} x_1 \prod_{t=m}^1 c_t^{-z_t} (c_1^{z_1} c_2^{z_2})^{-2}$,
6. $w = uv$, where $u = (c_1^{z_1} c_2^{z_2})^2,\ v \in E(m,n)$, and v is one of the following: $v = \prod_{t=1}^m c_t^{z_t} x_1^{-1} x_2^{-1}$ or $v = \prod_{t=1}^m c_t^{z_t} x_1^{-1} y_1^{-1}$.
7. $w = z_i v$.

Consider the first case. If $f_M(\sigma) = \mu_1 \cdots \mu_k$, and μ_1^α almost contains

$$x_i^{\phi_{K_1}} (A^*_{m+4i+K_2})^{-p_{m+4i+K_2}+1} x_{i+1}^{\phi_{K_2} \beta}$$

(which is a non-cancelled initial peace of $x_i^{2\phi_{K_1}\beta}$ up to a very short part of it), then either $T_{1,\text{left}}$ or $T_{2,\text{left}}$ is applicable and we split σ into two intervals σ_1 and σ_2 with labels $x_i^{2\phi_{K_1}}$ and $v^{\phi_{K_1}}$.

Suppose μ_1^α does not contain $x_i^{\phi_{K_1}}(A^*_{m+4i+K_2})^{-p_{m+4i+K_2}+1}x_{i+1}^{\phi_{K_2}\beta}$ up to a very short part. Then μ_k^α contains the non-cancelled left end E of $v^{\phi_{K+1}\beta}$, and $\mu_k^\alpha E^{-1}$ is not very short. In this case $T_{2,\text{right}}$ is applicable.

We can similarly consider all Cases 2-6.

Case 7. Letter z_i can appear only in the beginning of w (if z_i^{-1} appears at the end of w, we can replace w by w^{-1}) If $w = z_i t_1 \cdots t_s$ is the canonical decomposition, then $t_k = c_j^{\pm z_j}$ for each k. If μ_1^α is longer than the non-cancelled part of $(c_i^p z_i)^\beta$, or the difference between μ_1^α and $(c_i^p z_i)^\beta$ is very short, we can split σ into two parts, σ_1 with label $f_X(\sigma_1) = z^{\phi_{K_1}}$ and σ_2 with label $f_X(\sigma_2) = (t_1 \dots t_s)^{\phi_{K_1}}$.

If the difference between μ_1^α and $(c_i^p z_i)^\beta$ is not very short, and μ_1^α is shorter than the non-cancelled part of $(c_i^p z_i)^\beta$, then there is no interval δ with $f(\delta) \neq f(\sigma)$ such that $f_M(\delta)$ and $f_M(\sigma)$ end with μ_k, and we can split σ into two parts using T_1, T_2 and splitting μ_k.

We have considered all possible cases. □

Denote the resulting cut equation by Π'_{K_1}.

Corollary 7.10. *The intervals of Π'_{K_1} are labelled by elements $u^{\phi_{K_1}}$, where for $n = 1$*

$$u \in \{z_i,\ x_i,\ y_i,\ \prod c_s^{z_s},\ x_1 \prod_{t=m}^{1} c_t^{-z_t},\}$$

for $n = 2$

$$u \in \{z_i,\ x_i,\ y_i,\ \prod c_s^{z_s},\ y_1 x_1 \prod_{t=m}^{1} c_t^{-z_t},\ y_1 x_1,\ \prod_{t=1}^{m} c_t^{z_t} x_1 \prod_{t=m}^{1} c_t^{-z_t}\ ,\prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2^{\pm 1},$$
$$\prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2 x_1,\ \prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2 x_1 \prod_{t=m}^{1} c_t^{-z_t},\ x_1^{-1} x_2 x_1 \prod_{t=m}^{1} c_t^{-z_t},\ x_2 x_1 \prod_{t=m}^{1} c_t^{-z_t},$$
$$x_1^{-1} x_2,\ x_2 x_1\},$$

and for $n \geq 3$,

$$u \in \{z_i,\ x_i,\ y_i,\ c_s^{z_s},\ y_1 x_1 \prod_{t=m}^{3} c_t^{-z_t},\ \prod_{t=1}^{m} c_t^{z_t} x_1^{-1} x_2^{-1},\ y_r x_r,\ x_1 \prod_{t=m}^{1} c_t^{-z_t},$$
$$y_{r-2} x_{r-1}^{-1} x_r^{-1},\ y_{r-2} x_{r-1}^{-1},\ x_{r-1}^{-1} x_r^{-1},\ y_{r-1} x_r^{-1},\ r < n,\ x_{n-1}^{-1} x_n x_{n-1},$$
$$y_{n-2} x_{n-1}^{-1} x_n x_{n-1} y_{n-2}^{-1},\ y_{n-2} x_{n-1}^{-1} x_n^{\pm 1},\ x_{n-1}^{-1} x_n,\ x_n x_{n-1},$$
$$y_{n-1} x_n^{-1} x_{n-1} y_{n-2}^{-1},\ y_{n-1} x_n^{-1}, y_{r-1} x_r^{-1} y_r^{-1}\}.$$

Proof. Direct inspection from Lemma 7.9. □

Below we suppose $n > 0$. We still want to reduce the variety of possible labels of intervals in Π'_{K_1}. We cannot apply T_1, T_2 to some of the intervals labelled by $x^{\phi_{K_1}}$, $x \in X \cup E(m,n)$, because there are some cases when $x^{\phi_{K_1}}$ is completely cancelled in $y^{\phi_{K_1}}$, $x, y \in (X \cup E(m,n))^{\pm 1}$.

We will change the basis of $F(X \cup C_S)$, and then apply transformations T_1, T_2 to the labels written in the new basis. Replace, first, the basis $(X \cup C_S)$ by a new basis $\bar{X} \cup C_S$ obtained by replacing each variable x_s by $u_s = x_s y_{s-1}^{-1}$ for $s > 1$, and replacing x_1 by $u_1 = x_1 c_m^{-z_m}$.

Consider the case $n \geqslant 3$. Then the labels of the intervals will be rewritten as $u^{\phi_{K_1}}$, where

$$u \in \{z_i,\ u_i y_{i-1},\ y_i,\ \prod_s c_s^{z_s},\ y_1 u_1 \prod_{j=n-1}^{1} c_j^{-z_j},\ u_1^{-1} y_1^{-1} u_2^{-1},$$

$$y_r u_r y_{r-1},\ u_r,\ u_{r-1}^{-1} y_{r-1}^{-1} u_r^{-1},\ u_r y_{r-1} u_{r-1} y_{r-2},\ u_2 y_1 u_1 \prod_{j=n-1}^{1} c_j^{-z_j},\ r < n;$$

$$y_{n-2}^{-1} u_{n-1}^{-1} u_n y_{n-1} u_{n-1} y_{n-2},\ u_{n-1}^{-1} u_n y_{n-1} u_{n-1},\ u_{n-1}^{-1} u_n y_{n-1},$$

$$u_{n-1}^{-1} y_{n-1}^{-1} u_n^{-1}, y_{n-2}^{-1} u_{n-1}^{-1} u_n y_{n-1},\ u_n y_{n-1} u_{n-1} y_{n-2},\ u_n^{-1} u_{n-1},\ u_n\}.$$

In the cases $n = 1, 2$ some of the labels above do not appear, some coincide. Notice, that $x_n^{\phi_K} = u_n^{\phi_K} \circ y_{n-1}^{\phi_K}$, and that the first letter of $y_{n-1}^{\phi_K}$ is not cancelled in the products $(y_{n-1} x_{n-1} y_{n-2}^{-1})^{\phi_K}$, $(y_{n-1} x_{n-1})^{\phi_K}$ (see Lemma 4.8). Therefore, applying transformations similar to T_1 and T_2 to the cut equation Π'_{K_1} with labels written in the basis $\bar{X}$, we can split all the intervals with labels containing $(u_n y_{n-1})^{\phi_{K_1}}$ into two parts and obtain a cut equation with the same properties and intervals labelled by $u^{\phi_{K_1}}$, where

$$u \in \{z_i,\ u_i y_{i-1},\ y_i,\ \prod_s c_s^{z_s},\ y_1 u_1 \prod_{j=n-1}^{1} c_j^{-z_j},\ u_1^{-1} y_1^{-1} u_2^{-1},$$

$$y_r u_r y_{r-1}, u_r,\ u_{r-1}^{-1} y_{r-1}^{-1} u_r^{-1}, u_r y_{r-1} u_{r-1} y_{r-2},\ u_2 y_1 u_1 \prod_{j=n-1}^{1} c_j^{-z_j},\ r < n;$$

$$y_{n-2}^{-1} u_{n-1}^{-1} u_n,\ y_{n-1} u_{n-1} y_{n-2},\ u_{n-1}^{-1} u_n,\ y_{n-1} u_{n-1},\ u_n\}.$$

Consider for $i < n$ the expression for

$$(y_i u_i)^{\phi_K} = A_{m+4i}^{-p_{m+4i}+1} \circ x_{i+1} \circ A_{m+4i-4}^{-p_{m+4i-4}} \circ x^{p_{m+4i-3}} \circ y_i \circ A_{m+4i-2}^{p_{m+4i-2}-1} \circ x_i \circ \tilde{y}_{i-1}^{-1}.$$

Formula 3.a) from Lemma 4.21 shows that $u_i^{\phi_K}$ is completely cancelled in the product $y_i^{\phi_K} u_i^{\phi_K}$. This implies that $y_i^{\phi_K} = v_i^{\phi_K} \circ u_i^{-\phi_K}$.

Consider also the product

$$y_{i-1}^{-\phi_K} u_i^{-\phi_K} = \left(\mathbf{A}_{\mathbf{m+4i-4}}^{-\mathbf{p}_{\mathbf{m+4i-4}}+\mathbf{1}} \circ \mathbf{x}_\mathbf{i} \circ \tilde{\mathbf{y}}_{\mathbf{i-1}} \circ x_i^{-1} A_{m+4i-4}^{p_{m+4i-4}-1}\right) \left(A_{m+4i-4}^{-p_{m+4i-4}+1} x_i \circ (\mathbf{x}_\mathbf{i}^{\mathbf{p}_{\mathbf{m+4i-3}}} \mathbf{y}_{\mathbf{i-1}} \ldots *)^{\mathbf{p}_{\mathbf{m+4i-1}}-\mathbf{1}} \mathbf{x}_\mathbf{i}^{\mathbf{p}_{\mathbf{m+4i-3}}} \mathbf{y}_\mathbf{i} \mathbf{x}_{\mathbf{i+1}}^{-\mathbf{1}} \mathbf{A}_{\mathbf{m+4i}}^{\mathbf{p}_{\mathbf{m+4i}}-\mathbf{1}}\right),$$

where the non-cancelled part is made bold.

Notice that $(y_{r-1} u_{r-1})^{\phi_K} y_{r-2}^{\phi_K} = (y_{r-1} u_{r-1})^{\phi_K} \circ y_{r-2}^{\phi_K}$, because $u_{r-1}^{\phi_K}$ is completely cancelled in the product $y_i^{\phi_K} u_i^{\phi_K}$.

Therefore, we can again apply the transformations similar to T_1 and T_2 and split the intervals into the ones with labels $u^{\phi_{K_1}}$, where

$$u \in \{z_s,\ y_i,\ u_i,\ \prod_s c_s^{z_s},\ y_r u_r,\ y_1 u_1 \prod_{j=m-1}^{1} c_j^{-z_j},\ u_{n-1}^{-1} u_n = \bar{u}_n,$$
$$1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ 1 \leqslant r < n\}.$$

We change the basis again replacing $y_r, 1 < r < n$ by a new variable $v_r = y_r u_r$, and replacing $y_1 u_1 \prod_{j=m-1}^{1} c_j^{-z_j}$ by v_1. Then $y_r^{\phi_K} = v_r^{\phi_K} \circ u_r^{-\phi_K}$, and $y_1^{\phi_K} = v_1^{\phi_K} \circ c_1^{z_1^{\phi_K}} \circ c_{m-1}^{z_{m-1}^{\phi_K}} \circ u_1^{-\phi_K}$ (if $n \neq 1$). Formula 2.c) shows that $u_n^{\phi_K} = u_{n-1}^{\phi_K} \circ (u_{n-1}^{-1} u_n)^{\phi_K}$.

Apply transformations similar to T_1 and T_2 to the intervals with labels written in the new basis

$$\hat{X} = \{z_j,\ u_i,\ v_i,\ y_n,\ \bar{u}_n = u_{n-1} u_n,\ 1 \leqslant j \leqslant m,\ 1 \leqslant i < n,\ j \leqslant m\},$$

and obtain intervals with labels $u^{\phi_{K_1}}$, where

$$u \in \hat{X} \cup \{c_m^{z_m}\}.$$

Denote the resulting cut equation by $\bar{\Pi}_{K_1} = (\bar{\mathcal{E}}, f_{\bar{X}}, f_{\bar{M}})$. Let α be the corresponding solution of $\bar{\Pi}_{K_1}$ with respect to β.

Denote by $\bar{M}_{\text{side}}$ the set of long variables in $\bar{\Pi}_{K_1}$, then $\bar{M} = \bar{M}_{\text{veryshort}} \cup \bar{M}_{\text{side}}$.

Define a binary relation $\sim_{\text{left}}$ on $\bar{M}_{\text{side}}^{\pm 1}$ as follows. For $\mu_1, \mu_1' \in \bar{M}_{\text{side}}^{\pm 1}$ put $\mu_1 \sim_{\text{left}} \mu_1'$ if and only if there exist two intervals $\sigma, \sigma' \in \bar{E}$ with $f_{\bar{X}}(\sigma) = f_{\bar{X}}(\sigma')$ such that

$$f_{\bar{M}}(\sigma) = \mu_1 \mu_2 \cdots \mu_r, \quad f_{\bar{M}}(\sigma') = \mu_1' \mu_2' \cdots \mu_{r'}'$$

and either $\mu_r = \mu_{r'}'$ or $\mu_r, \mu_{r'}' \in M_{\text{veryshort}}$. Observe that if $\mu_1 \sim_{\text{left}} \mu_1'$ then

$$\mu_1 = \mu_1' \lambda_1 \cdots \lambda_t$$

for some $\lambda_1, \ldots, \lambda_t \in M_{\text{veryshort}}^{\pm 1}$. Notice, that $\mu \sim_{\text{left}} \mu$.

Similarly, we define a binary relation $\sim_{\text{right}}$ on $\bar{M}_{\text{side}}^{\pm 1}$. For $\mu_r, \mu_{r'}' \in \bar{M}_{\text{side}}^{\pm 1}$ put $\mu_r \sim_{\text{right}} \mu_{r'}'$ if and only if there exist two intervals $\sigma, \sigma' \in \bar{E}$ with $f_{\bar{X}}(\sigma) = f_{\bar{X}}(\sigma')$ such that

$$f_{\bar{M}}(\sigma) = \mu_1 \mu_2 \cdots \mu_r, \quad f_{\bar{M}}(\sigma') = \mu_1' \mu_2' \cdots \mu_{r'}'$$

and either $\mu_1 = \mu_1'$ or $\mu_1, \mu_1' \in M_{\text{veryshort}}$. Again, if $\mu_r \sim_{\text{right}} \mu_{r'}'$ then

$$\mu_r = \lambda_1 \ldots \lambda_t \mu_{r'}'$$

for some $\lambda_1, \ldots, \lambda_t \in M_{\text{veryshort}}^{\pm 1}$.

Denote by $\sim$ the transitive closure of

$$\{(\mu, \mu') \mid \mu \sim_{\text{left}} \mu'\} \cup \{(\mu, \mu') \mid \mu \sim_{\text{right}} \mu'\} \cup \{(\mu, \mu^{-1}) \mid \mu \in \bar{M}_{\text{side}}^{\pm 1}\}.$$

Clearly, $\sim$ is an equivalence relation on $\bar{M}_{\text{side}}^{\pm 1}$. Moreover, $\mu \sim \mu'$ if and only if there exists a sequence of variables

$$\mu = \mu_0, \mu_1, \ldots, \mu_k = \mu' \tag{43}$$

from $\bar{M}_{\text{side}}^{\pm 1}$ such that either $\mu_{i-1} = \mu_i$, or $\mu_{i-1} = \mu_i^{-1}$, or $\mu_{i-1} \sim_{\text{left}} \mu_i$, or $\mu_{i-1} \sim_{\text{right}} \mu_i$ for $i = 1, \ldots, k$. Observe that if μ_{i-1} and μ_i from (43) are side variables of "different sides" (one is on the left, and the other is on the right) then

$\mu_i = \mu_{i-1}^{-1}$. This implies that replacing in the sequence (43) some elements μ_i with their inverses one can get a new sequence

$$\mu = \nu_0, \nu_1, \ldots, \nu_k = (\mu')^{\varepsilon} \tag{44}$$

for some $\varepsilon \in \{1, -1\}$ where $\nu_{i-1} \sim \nu_i$ and all the variables ν_i are of the same side. It follows that if μ is a left-side variable and $\mu \sim \mu'$ then

$$(\mu')^{\varepsilon} = \mu \lambda_1 \cdots \lambda_t \tag{45}$$

for some $\lambda_j \in M_{\mathrm{veryshort}}^{\pm 1}$.

It follows from (45) that for a variable $\nu \in \bar{M}_{\mathrm{side}}^{\pm 1}$ all variables from the equivalence class $[\nu]$ of ν can be expressed via ν and very short variables from $M_{\mathrm{veryshort}}$. So if we fix a system of representatives R of $\bar{M}_{\mathrm{side}}^{\pm 1}$ relative to $\sim$ then all other variables from $\bar{M}_{\mathrm{side}}$ can be expressed as in (45) via variables from R and very short variables.

This allows one to introduce a new transformation T_3 of cut equations. Namely, if a set of representatives R is fixed then using (45) replace every variable ν in every word $f_M(\sigma)$ of a cut equation Π by its expression via the corresponding representative variable from R and a product of very short variables.

Now we repeatedly apply the transformation T_3 till the equivalence relations $\sim_{\mathrm{left}}$ and $\sim_{\mathrm{right}}$ become trivial. This process stops in finitely many steps since the non-trivial relations decrease the number of side variables.

Denote the resulting equation again by $\bar{\Pi}_{K_1}$.

Now we introduce an equivalence relation on partitions of $\bar{\Pi}_{K_1}$. Two partitions $f_M(\sigma)$ and $f_M(\delta)$ are equivalent ($f_M(\sigma) \sim f_M(\delta)$) if $f_X(\sigma) = f_X(\delta)$ and either the left side variables or the right side variables of $f_M(\sigma)$ and $f_M(\delta)$ are equivalent. Observe, that $f_X(\sigma) = f_X(\delta)$ implies $f_M(\sigma)^{\alpha} = f_M(\delta)^{\alpha}$, so in this case the partitions $f_M(\sigma)$ and $f_M(\delta)$ cannot begin with μ and μ^{-1} correspondingly. It follows that if $f_M(\sigma) \sim f_M(\delta)$ then the left side variables and, correspondingly, the right side variables of $f_M(\sigma)$ and $f_M(\delta)$ (if they exist) are equal. Therefore, the relation $\sim$ is, indeed, an equivalence relation on the set of partitions of $\bar{\Pi}_{K_1}$.

If an equivalence class of partitions contains two distinct elements $f_M(\sigma)$ and $f_M(\delta)$ then the equality

$$f_M(\sigma)^{\alpha} = f_M(\delta)^{\alpha}$$

implies the corresponding equation on the variables $\bar{M}_{\mathrm{veryshort}}$, which is obtained by deleting all side variables (which are equal) from $f_M(\sigma)$ and $f_M(\delta)$ and equalizing the resulting words in very short variables. Denote by $\Delta(\bar{M}_{\mathrm{veryshort}}) = 1$ this system.

Now we describe a transformation T_4. Fix a set of representatives R_p of partitions of $\bar{\Pi}_{K_1}$ with respect to the equivalence relation $\sim$. For a given class of equivalent partitions we take as a representative an interval σ with $f_M(\sigma) = \mu_{\mathrm{left}} \ldots \mu_{\mathrm{right}}$. Below we say that μ^{α} almost contains u^{β} if μ^{α} contains a subword which is the reduced form of $c_1 u^{\beta} c_2$ for some $c_1, c_2 \in C_{\beta}$.

Principal variables A long variable μ_{left} or μ_{right} for the interval σ which represents a class of equivalent partitions is called *principal* **in** σ in the following cases.

1) Let $f_X(\sigma) = u_i$ $(i \neq n)$, where $u_i = x_i y_{i-1}^{-1}$ for $i > 1$ and $u_1 = x_1 c_m^{-z_m}$ for $m \neq 0$. Then (see Lemma 4.21)

$$u_i^{\phi_{K_1}} = A^{*-q_4+1}_{K_2+m+4i} x_{i+1}^{\phi_{K_2}} y_i^{-\phi_{K_2}} x_i^{-q_1\phi_{K_2}}$$
$$\left(x_i^{-\phi_{K_2}} A^{*q_0}_{K_2+m+4i-4} A^{*(-q_2+1)}_{K_2+m+4i-2} y_i^{\phi_{K_2}} x_i^{-q_1\phi_{K_2}}\right)^{q_3-1} A^{*q_0}_{K_2+m+4i-4}.$$

A right variable μ_{right} is principal in σ if $\mu^{\alpha}_{\text{right}}$ almost contains a cyclically reduced part of

$$\left(x_i^{-\psi_{K_2}} A^{*q_0\beta}_{K_2+m+4i-4} A^{*(-q_2+1)\beta}_{m+4i-2} y_i^{\psi_{K_2}} x_i^{-q_1\psi_{K_2}}\right)^q$$
$$= (x_i^{q_1} y_i)^{\psi_{K_2}} (A^{*\beta}_{K_2+m+4i-1})^{-q} (y_i^{-1} x_i^{-q_1})^{\psi_{K_2}},$$

for some $q > 2$. If μ_{right} is not principal, thenwe define μ_{left} as principal.

2) Let $f_X(\sigma) = v_i$, where $v_i = y_i u_i$ $(i \neq 1, n)$ and $v_1 = y_1 u_1 \prod_{j=m-1}^{1} c_j^{-z_j}$. Then (see formula 3.a) from Lemma 4.21)

$$v_i^{\phi_{K_1}} = A^{*(-q_4+1)}_{K_2+m+4i} x_{i+1}^{\phi_{K_2}} A^{*(-q_0)}_{K_2+m+4i-4} x_i^{q_1\phi_{K_2}} y_i^{\phi_{K_2}} A^{*(q_2-1)}_{K_2+m+4i-2} A^{*-1}_{K_2+m+4i-4},$$

if $i \neq 1$, and

$$v_1^{\phi_{K_1}} = A^{*(-q_4+1)}_{K_2+m+4} x_2^{\phi_{K_2}} A^{*(-q_0)}_{K_2+2m} x_1^{q_1\phi_{K_2}} y_1^{\phi_{K_2}} A^{*(q_2-1)}_{K_2+m+1} x_1 \Pi_{j=n}^{1} c_j^{-z_j},$$

if $i = 1$.
A side variable μ_{right} or μ_{left} is principal if $\mu^{\alpha}_{\text{right}}$ (correspondingly, $\mu^{\alpha}_{\text{left}}$) almost contains $(A^{\beta}_{K_2+m+4i})^{-q}$, for some $q > 2$.

3) Let $f_X(\sigma) = \bar{u}_n$. Formula 3.c) from Lemma 4.21 gives $\bar{u}_n^{\phi_{K_1}} = A^{*}_{K_2+m+4n-8}$
$A^{-q_2+1}_{K_2+m+4n-6} (y_{n-1}^{-1} x_n^{-q_1})^{\phi_{K_1}} A^{*q_0}_{K_2+m+4n-8} (x_n^{q_5} y_n)^{\phi_{K_1}} A^{*q_6-1}_{K_2+m+4n-2} A^{*-1}_{K_2+m+4n-4}$.

A side variable μ_{right} or μ_{left} is principal if $\mu^{\alpha}_{\text{right}}$ (correspondingly, $\mu^{\alpha}_{\text{left}}$) almost contains $(A^{\beta}_{K_2+m+4n-2})^q$, $q > 2$.

4) Let $f_X(\sigma) = y_n$. A side variable μ_{right} or μ_{left} is principal if $\mu^{\alpha}_{\text{right}}$ $(\mu^{\alpha}_{\text{left}})$ almost contains $(A^{\beta}_{K_2+m+4n-1})^q$, $2q > p_{K_1} - 2$.

5) Let $f_X(\sigma) = z_j$, $j = 1, \ldots, m-1$. Then (by Lemma 4.6)

$$z_j^{\phi_{K_1}} = c_j z_j^{\phi_{K_2}} A^{*\beta p_{j-1}}_{K_2+j-1} c_{j+1}^{z_{j+1}^{\phi_{K_2}}} A^{*\beta p_j - 1}_{K_2+j}.$$

A variable μ_{left} (μ_{right}) is *principal* if $\mu^{\alpha}_{\text{right}}$ (correspondingly, $\mu^{\alpha}_{\text{left}}$) almost contains $(A^{\beta}_{K_2+j})^q$, for some $|q| > 2$. Both left and right side variables can be simultaneously principal.

6) Let $f_X(\sigma) = z_m$. Then $z_m^{\phi_{K_1}} = c_m^{K_2} z_m^{\phi_{K_2}} A^{*p_{m-1}}_{K_2+m-1} x_1^{-\phi_{K_2}} A^{*p_m-1}_{K_2+m}$.
In this case μ_{left} is *principal* in σ if and only if μ_{left} is long (i.e., it is not very short), and we define μ_{right} to be always non-principal. Observe that if μ_{left} is very short then $\mu^{\alpha}_{\text{right}} = f z_m^{\phi_{K_1}\beta}$ for a very short $f \in F$.

Let $f_X(\sigma) = z_m^{-1} c_m z_m$. By Lemma 4.6 $f_X(\sigma)^{\phi_{K_1}} = A^{*-p_m+1}_{K_2+m} x_1^{\phi_{K_2}} A^{*p_m}_{K_2+m}$.

The variable μ_{left} is *principal* in σ if and only if the following two conditions hold: $\mu_{\text{left}}^{\alpha}$ almost contains $(A_{K_2+m}^{\beta})^q$, for some q with $|q| > 2$; $\mu_{\text{left}}^{-1} \neq f z_m^{\phi_{K_1}\beta}$ for a very short $f \in F$.

Similarly, the variable μ_{right} is *principal* in σ if and only if the following two conditions hold: $\mu_{\text{right}}^{\alpha}$ almost contains $(A_{K_2+m}^{\beta})^q$, for some q with $|q| > 2$; $\mu_{\text{right}}^{\alpha} \neq f z_m^{\phi_{K_1}\beta}$ for a very short $f \in F$.

Observe, that in this case the variables μ_{left} and μ_{right} can be simultaneously principal in σ and non-principal in σ. The latter happens if and only if $\mu_{\text{right}}^{\alpha} = f_1 z_m^{\phi_{K_1}\beta}$ and $\mu_{\text{left}}^{\alpha} = z_m^{-\phi_{K_1}\beta} f_2$ for some very short elements $f_1, f_2 \in F$. Therefore, if both μ_{left} and μ_{right} are non-principal then they can be expressed in terms of $z_m^{\phi_{K_1}}$ and very short variables.

CLAIM 18. *Every partition has at least one principal variable, unless this partition is of that particular type from Case 6).*

CLAIM 19. *If both side variables of a partition of $\bar{\Pi}_{K_1}$ are non-principal, then they are non-principal in every partition of $\bar{\Pi}_{K_1}$.*

CLAIM 20. *Let $n \neq 0$. Then a side variable can be principal only in one class of equivalent partitions.*

PROOF. Follows from the definition of principal variables. $\square$

For the cut equation $\bar{\Pi}_{K_1}$ we construct a finite graph $\Gamma = (V, E)$. Every vertex from V is marked by variables from $\bar{M}_{\text{side}}^{\pm 1}$ and letters from the alphabet $\{P, N\}$. Every edge from E is colored either as red or blue. The graph Γ is constructed as follows. Every partition $f_M(\sigma) = \mu_1 \cdots \mu_k$ of $\bar{\Pi}_{K_1}$ gives two vertices $v_{\sigma,\text{left}}$ and $v_{\sigma,\text{right}}$ into Γ, so

$$V = \bigcup_{\sigma} \{v_{\sigma,\text{left}}, v_{\sigma,\text{right}}\}.$$

We mark $v_{\sigma,\text{left}}$ by μ_1 and $v_{\sigma,\text{right}}$ by μ_k. Now we mark the vertex $v_{\sigma,\text{left}}$ by a letter P or letter N if μ_1 is correspondingly principal or non-principal in σ. Similarly, we mark $v_{\sigma,\text{right}}$ by P or N if μ_k is principal or non-principal in σ.

For every σ the vertices $v_{\sigma,left}$ and $v_{\sigma,right}$ are connected by a *red* edge. Also, we connect by a *blue* edge every pair of vertices which are marked by variables μ, ν provided $\mu = \nu$ or $\mu = \nu^{-1}$. This describes the graph Γ.

Below we construct a new graph Δ which is obtained from Γ by deleting some blue edges according to the following procedure. Let B be a maximal connected blue component of Γ, i.e., a connected component of the graph obtained from Γ by deleting all red edges. Notice, that B is a complete graph, so every two vertices in B are connected by a blue edge. Fix a vertex v in B and consider the star-subgraph $Star_B$ of B generated by all edges adjacent to v. If B contains a vertex marked by P then we choose v with label P, otherwise v is an arbitrary vertex of B. Now, replace B in Γ by the graph $Star_B$, i.e., delete all edges in B which are not adjacent to v. Repeat this procedure for every maximal blue component B of Γ. If the blue component corresponds to long bases of case 6) that are non-principal and equal to $f_1 z_m^{\phi_{K_1}} f_2$ for very short f_1, f_2, we remove all the blue edges that produce cycles if the red edge from Γ connecting non-principal μ_{left} and μ_{right} is added to the component (if such a red edge exists). Denote the resulting graph by Δ.

In the next claim we describe connected components of the graph Δ.

Claim 21. *Let C be a connected component of Δ. Then one of the following holds:*

(1) *there is a vertex in C marked by a variable which does not occur as a principal variable in any partition of $\bar{\Pi}_{K_1}$. In particular, any component which satisfies one of the following conditions has such a vertex:*
   a) *there is a vertex in C marked by a variable which is a short variable in some partition of $\bar{\Pi}_{K_1}$.*
   b) *there is a red edge in C with both endpoints marked by N (it corresponds to a partition described in Case 6 above);*

(2) *both endpoints of every red edge in C are marked by P. In this case C is an isolated vertex;*

(3) *there is a vertex in C marked by a variable μ and N and if μ occurs as a label of an endpoint of some red edge in C then the other endpoint of this edge is marked by P.*

Proof. Let C be a connected component of Δ. Observe first, that if μ is a short variable in $\bar{\Pi}_{K_1}$ then μ is not principle in σ for any interval σ from $\bar{\Pi}_{K_1}$, so there is no vertex in C marked by both μ and P. Also, it follows from Claim 19 that if there is a red edge e in C with both endpoints marked by N, then the variables assigned to endpoints of e are non-principle in any interval σ of $\bar{\Pi}_{K_1}$. This proves the part "in particular" of 1).

Now assume that the component C does not satisfy any of the conditions (1), (2). We need to show that C has type (3). It follows that every variable which occurs as a label of a vertex in C is long and it labels, at least, one vertex in C with label P. Moreover, there are non-principle occurrences of variables in C.

We summarize some properties of C below:

- There are no blue edges in Δ between vertices with labels N and N (by construction).
- There are no blue edges between vertices labelled by P and P (Claim 20).
- There are no red edges in C between vertices labelled by N and N (otherwise 1) would hold).
- Any reduced path in Δ consists of edges of alternating color (by construction).

We claim that C is a tree. Let $p = e_1 \dots e_k$ be a simple loop in C (every vertex in p has degree 2 and the terminal vertex of e_k is equal to the starting point of e_1).

We show first that p does not have red edges with endpoints labelled by P and P. Indeed, suppose there exists such an edge in p. Taking cyclic permutation of p we may assume that e_1 is a red edge with labels P and P. Then e_2 goes from a vertex with label P to a vertex with label N. Hence the next red edge e_3 goes from N to P, etc. This shows that every blue edge along p goes from P to N. Hence the last edge e_k which must be blue goes from P to N -contradiction, since all the labels of e_1 are P.

It follows that both colors of edges and labels of vertices in p alternate. We may assume now that p starts with a vertex with label N and the first edge e_1 is red. It follows that the end point of e_1 is labelled by N and all blue edges go from N to P. Let e_i be a blue edge from v_i to v_{i+1}. Then the variable μ_i assign to the vertex v_i is principal in the partition associated with the red edge e_{i-1} , and the variable $\mu_{i+1} = \mu_i^{\pm 1}$ associated with v_{i+1} is a non-principal side variable in the

partition $f_M(\sigma)$ associated with the red edge e_{i+1}. Therefore, the the side variable μ_{i+2} associated with the end vertex v_{i+2} is a principal side variable in the partition $f_M(\sigma)$ associated with e_{i+1}. It follows from the definition of principal variables that the length of μ_{i+2}^α is much longer than the length of μ_{i+1}^α, unless the variable μ_i is described in the Case 1). However, in the letter case the variable μ_{i+2} cannot occur in any other partition $f_M(\delta)$ for $\delta \neq \sigma$. This shows that there no blue edges in Δ with endpoints labelled by such μ_{i+2}. This implies that v_{i+2} has degree one in Δ - contradiction wit the choice of p. This shows that there are no vertices labelled by such variables described in Case 1). Notice also, that the length of variables (under α) is preserved along blue edges: $|\mu_{i+1}^\alpha| = |(\mu_i^{\pm 1})^\alpha| = |\mu_i^\alpha|$. Therefore,

$$|\mu_i^\alpha| = |\mu_{i+1}^\alpha| < |\mu_{i+2}^\alpha|$$

for every i.

It follows that going along p the length of $|\mu_i^\alpha|$ increases, so p cannot be a loop. This implies that C is a tree.

Now we are ready to show that the component C has type (3). Let μ_1 be a variable assigned to some vertex v_1 in C with label N. If μ_1 satisfies the condition (3) then we are done. Otherwise, μ_1 occurs as a label of one of P-endpoints, say v_2 of a red edge e_2 in C such that the other endpoint of e_2, say v_3 is non-principal. Let μ_3 be the label of v_3. Thus v_1 is connected to v_2 by a blue edge and v_2 is connected to v_3 by a red edge. If μ_3 does not satisfy the condition (3) then we can repeat the process (with μ_3 in place of μ_1). The graph C is finite, so in finitely many steps either we will find a variable that satisfies (3) or we will construct a closed reduced path in C. Since C is a tree the latter does not happen, therefore C satisfies (3), as required.

□

Claim 22. *The graph Δ is a forest, i.e., it is union of trees.*

Proof. Let C be a connected component of Δ. If C has type (3) then it is a tree, as has been shown in Claim 21 If C of the type (2) then by Claim 21 C is an isolated vertex – hence a tree. If C is of the type (1) then C is a tree because each interval corresponding to this component has exactly one principal variable, and the same long variable cannot be principal in two different intervals. Although the same argument as in (3) also works here.

□

Now we define the sets $\bar{M}_{\text{useless}}, \bar{M}_{\text{free}}$ and assign values to variables from $\bar{M} = \bar{M}_{\text{useless}} \cup \bar{M}_{free} \cup \bar{M}_{\text{veryshort}}$. To do this we use the structure of connected components of Δ. Observe first, that all occurrences of a given variable from $\bar{M}_{\text{sides}}$ are located in the same connected component.

Denote by $\bar{M}_{free}$ subset of $\bar{M}$ which consists of variables of the following types:

(1) variables which do not occur as principal in any partition of $(\bar{\Pi}_{K_1})$;
(2) one (but not the other) of the variables μ and ν if they are both principal side variables of a partition of the type (21) and such that $\nu \neq \mu^{-1}$.

Denote by $\bar{M}_{\text{useless}} = \bar{M}_{\text{side}} - \bar{M}_{\text{free}}$.

Claim 23. *For every $\mu \in \bar{M}_{\text{useless}}$ there exists a word*

$$V_\mu \in F[X \cup \bar{M}_{\text{free}} \cup \bar{M}_{\text{veryshort}}]$$

such that for every map $\alpha_{\text{free}} : \bar{M}_{\text{free}} \to F$, and every solution

$$\alpha_s : F[\bar{M}_{\text{veryshort}}] \to F$$

of the system $\Delta(\bar{M}_{\text{veryshort}}) = 1$ the map $\alpha : F[\bar{M}] \to F$ defined by

$$\mu^\alpha = \begin{cases} \mu^{\alpha_{\text{free}}} & \text{if } \mu \in \bar{M}_{\text{free}}; \\ \mu^{\alpha_s} & \text{if } \mu \in \bar{M}_{\text{veryshort}}; \\ \bar{V}_\mu(X^\delta, \bar{M}_{\text{free}}^{\alpha_{\text{free}}}, \bar{M}_{\text{veryshort}}^{\alpha_s}) & \text{if } \mu \in \bar{M}_{\text{useless}}. \end{cases}$$

is a group solution of $\bar{\Pi}_{K_1}$ with respect to β.

Proof. The claim follows from Claims 21 and 22. Indeed, take as values of short variables an arbitrary solution α_s of the system $\Delta(\bar{M}_{\text{veryshort}}) = 1$. This system is obviously consistent, and we fix its solution. Consider connected components of type (1) in Claim 21. If μ is a principal variable for some σ in such a component, we express μ^α in terms of values of very short variables $\bar{M}_{\text{veryshort}}$ and elements $t^{\psi_{K_1}}$, $t \in X$ that correspond to labels of the intervals. This expression does not depend on α_s, β and tuples q, p^*. For connected components of Δ of types (2) and (3) we express values μ^α for $\mu \in M_{\text{useless}}$ in terms of values ν^α, $\nu \in M_{\text{free}}$ and $t^{\psi_{K_1}}$ corresponding to the labels of the intervals. □

We can now finish the proof of Proposition 7.8. Observe, that $M_{\text{veryshort}} \subseteq \bar{M}_{\text{veryshort}}$. If λ is an additional very short variable from $M^*_{\text{veryshort}}$ that appears when transformation T_1 or T_2 is performed, λ^α can be expressed in terms $M^\alpha_{\text{veryshort}}$. Also, if a variable λ belongs to $\bar{M}_{\text{free}}$ and does not belong to M, then there exists a variable $\mu \in M$, such that $\mu^\alpha = u^{\psi_{K_1}} \lambda^\alpha$, where $u \in F(X, C_S)$, and we can place μ into M_{free}.

Observe, that the argument above is based only on the tuple p, it does not depend on the tuples p^* and q. Hence the words V_μ do not depend on p^* and q.

The Proposition is proved for $n \neq 0$. If $n = 0$, partitions of the intervals with labels $z_{n-1}^{\phi_{K_1}}$ and $z_n^{\phi_{K_1}}$ can have equivalent principal right variables, but in this case the left variables will be different and do not appear in other non-equivalent partitions. The connected component of Δ containing these partitions will have only four vertices one blue edge.

In the case $n = 0$ we transform equation Π_{K_1} applying transformation T_1 to the form when the intervals are labelled by $u^{\phi_{K_1}}$, where

$$u \in \left\{ z_1, \ldots, z_m, c_{m-1}^{z_{m-1}}, z_m c_{m-1}^{-z_{m-1}} \right\}.$$

If μ_{left} is very short for the interval δ labelled by $(z_m c_{m-1}^{-z_{m-1}})^{\phi_{K_1}}$, we can apply T_2 to δ, and split it into intervals with labels $z_m^{\phi_{K_1}}$ and $c_{m-1}^{-z_{m-1}^{\phi_{K_1}}}$. Indeed, even if we had to replace μ_{right} by the product of two variables, the first of them would be very short.

If μ_{left} is not very short for the interval δ labelled by

$$(z_m c_{m-1}^{-z_{m-1}})^{\phi_{K_1}} = c_m z_m^{\phi_{K_2}} A_{m-1}^{*p_{m-1}-1},$$

we do not split the interval, and μ_{left} will be considered as the principal variable for it. If μ_{left} is not very short for the interval δ labelled by $z_m^{\phi_{K_1}} = z_m^{\phi_{K_2}} A_{m-1}^{*p_{m-1}}$, it is a principal variable, otherwise μ_{right} is principal.

If an interval δ is labelled by $(c_{m-1}^{z_{m-1}})^{\phi_{K_1}} = A_{m-1}^{*-p_{m-1}+1} c_m^{-z_m^{\phi_{K_2}}} A_{m-1}^{*p_{m-1}}$, we consider μ_{right} principal if $\mu_{\text{right}}^{\alpha}$ ends with $(c_m^{-z_m^{\phi_{K_2}}} A_{m-1}^{*p_{m-2}})^{\beta}$, and the difference is not very short. If $\mu_{\text{left}}^{\alpha}$ is almost $z_m^{-\phi_k\beta}$ and $\mu_{\text{right}}^{\alpha}$ is almost $z_m^{\phi_k\beta}$, we do not call any of the side variables principal. In all other cases μ_{left} is principal.

Definition of the principal variable in the interval with label $z_i^{\phi_{K_1}}$, $i = 1, \dots, m-2$ is the same as in 5) for $n \neq 0$.

A variable can be principal only in one class of equivalent partitions. All the rest of the proof is the same as for $n > 0$.

$\square$

Now we continue the proof of Theorem A. Let $L = 2K + \kappa(\Pi)4K$ and

$$\Pi_\phi = \Pi_L \to \Pi_{L-1} \to \dots \to \dots$$

be the sequence of Γ-cut equations (42). For a Γ-cut equation Π_j from (42) by M_j and α_j we denote the corresponding set of variables and the solution relative to β.

By Claim 10 in the sequence (42) either there is $3K$-stabilization at $K(r+2)$ or $Comp(\Pi_{K(r+1)}) = 0$.

Case 1. Suppose there is $3K$-stabilization at $K(r+2)$ in the sequence (42).

By Proposition 7.8 the set of variables $M_{K(r+1)}$ of the cut equation $\Pi_{K(r+1)}$ can be partitioned into three subsets

$$M_{K(r+1)} = M_{\text{veryshort}} \cup M_{\text{free}} \cup M_{\text{useless}}$$

such that there exists a finite consistent system of equations $\Delta(M_{\text{veryshort}}) = 1$ over F and words $V_\mu \in F[X, M_{\text{free}}, M_{\text{veryshort}}]$, where $\mu \in M_{\text{useless}}$, such that for every solution $\delta \in \mathcal{B}$, for every map $\alpha_{\text{free}} : M_{\text{free}} \to F$, and every solution $\alpha_{short} : F[M_{\text{veryshort}}] \to F$ of the system $\Delta(M_{\text{veryshort}}) = 1$ the map $\alpha_{K(r+1)} : F[M] \to F$ defined by

$$\mu^{\alpha_{K(r+1)}} = \begin{cases} \mu^{\alpha_{\text{free}}} & \text{if } \mu \in M_{\text{free}} \\ \mu^{\alpha_{\text{short}}} & \text{if } \mu \in M_{\text{veryshort}} \\ V_\mu(X^\delta, M_{\text{free}}^{\alpha_{\text{free}}}, M_{\text{veryshort}}^{\alpha_s}) & \text{if } \mu \in M_{\text{useless}} \end{cases}$$

is a group solution of $\Pi_{K(r+1)}$ with respect to β. Moreover, the words V_μ do not depend on tuples p^* and q.

By Claim 3 if $\Pi = (\mathcal{E}, f_X, f_M)$ is a Γ-cut equation and $\mu \in M$ then there exists a word $\mathcal{M}_\mu(M_{T(\Pi)}, X)$ in the free group $F[M_{T(\Pi)} \cup X]$ such that

$$\mu^{\alpha_\Pi} = \mathcal{M}_\mu\left(M_{T(\Pi)}^{\alpha_{T(\Pi)}}, X^{\phi_{K(r+1)}}\right)^\beta,$$

where α_Π and $\alpha_{T(\Pi)}$ are the corresponding solutions of Π and $T(\Pi)$ relative to β.

Now, going along the sequence (42) from $\Pi_{K(r+1)}$ back to the cut equation Π_L and using repeatedly the remark above for each $\mu \in M_L$ we obtain a word

$$\mathcal{M}'_{\mu,L}(M_{K(r+1)}, X^{\phi_{K(r+1)}}) = \mathcal{M}'_{\mu,L}(M_{\text{useless}}, M_{\text{free}}, M_{\text{veryshort}}, X^{\phi_{K(r+1)}})$$

such that

$$\mu^{\alpha_L} = \mathcal{M}'_{\mu,L}(M_{K(r+1)}^{\alpha_{K(r+1)}}, X^{\phi_{K(r+1)}})^\beta.$$

Let $\delta = \phi_{K(r+1)} \in \mathcal{B}$ and put

$$\begin{aligned}&\mathcal{M}_{\mu,L}(X^{\phi_{K(r+1)}})\\ &\quad = \mathcal{M}'_{\mu,L}(V_\mu(X^{\phi_{K(r+1)}}, M_{\text{free}}^{\alpha_{\text{free}}}, M_{\text{veryshort}}^{\alpha_{short}}), M_{\text{free}}^{\alpha_{\text{free}}}, M_{\text{veryshort}}^{\alpha_{short}}, X^{\phi_{K(r+1)}}).\end{aligned}$$

Then for every $\mu \in M_L$

$$\mu^{\alpha_L} = \mathcal{M}_{\mu,L}(X^{\phi_{K(r+1)}})^{\beta}$$

If we denote by $\mathcal{M}_L(X)$ a tuple of words

$$\mathcal{M}_L(X) = (\mathcal{M}_{\mu_1,L}(X), \ldots, \mathcal{M}_{\mu_{|M_L|},L}(X)),$$

where $\mu_1, \ldots, \mu_{|M_L|}$ is some fixed ordering of M_L then

$$M_L^{\alpha_L} = \mathcal{M}_L(X^{\phi_{K(r+1)}})^{\beta}.$$

Observe, that the words $\mathcal{M}_{\mu,L}(X)$, hence $\mathcal{M}_L(X)$ (where $X^{\phi_{K(r+1)}}$ is replaced by X) are the same for every $\phi_L \in \mathcal{B}_{p,q}$.

It follows from property c) of the cut equation Π_ϕ that the solution α_L of Π_ϕ with respect to β gives rise to a group solution of the original cut equation $\Pi_{\mathcal{L}}$ with respect to $\phi_L \circ \beta$.

Now, property c) of the initial cut equation $\Pi_{\mathcal{L}} = (\mathcal{E}, f_X, f_{M_L})$ insures that for every $\phi_L \in \mathcal{B}_{p,q}$ the pair $(U_{\phi_L\beta}, V_{\phi_L\beta})$ defined by

$$U_{\phi_L\beta} = Q(M_L^{\alpha_L}) = Q(\mathcal{M}_L(X^{\phi_{K(r+1)}}))^{\beta},$$

$$V_{\phi_L\beta} = P(M_L^{\alpha_L}) = P(\mathcal{M}_L(X^{\phi_{K(r+1)}}))^{\beta}.$$

is a solution of the system$S(X) = 1 \wedge T(X,Y) = 1$.

We claim that

$$Y(X) = P(\mathcal{M}_L(X))$$

is a solution of the equation $T(X,Y) = 1$ in $F_{R(S)}$. By Theorem 5.3 $\mathcal{B}_{p,q,\beta}$ is a discriminating family of solutions for the group $F_{R(S)}$. Since

$$T(X, Y(X))^{\phi\beta} = T(X^{\phi\beta}, Y(X^{\phi\beta})) = T(X^{\phi\beta}, \mathcal{M}_L(X^{\phi\beta})) = T(U_{\phi_L\beta}, V_{\phi_L\beta}) = 1$$

for any $\phi\beta \in \mathcal{B}_{p,q,\beta}$ we deduce that $T(X, Y_{p,q}(X)) = 1$ in $F_{R(S)}$.

Now we need to show that $T(X,Y) = 1$ admits a complete S-lift. Let $W(X,Y) \neq 1$ be an inequality such that $T(X,Y) = 1 \wedge W(X,Y) \neq 1$ is compatible with $S(X) = 1$. In this event, one may assume (repeating the argument from the beginning of this section) that the set

$$\Lambda = \{(U_\psi, V_\psi) \mid \psi \in \mathcal{L}_2\}$$

is such that every pair $(U_\psi, V_\psi) \in \Lambda$ satisfies the formula $T(X,Y) = 1 \wedge W(X,Y) \neq 1$. In this case, $W(X, Y_{p,q}(X)) \neq 1$ in $F_{R(S)}$, because its image in F is non-trivial:

$$W(X, Y_{p,q}(X))^{\phi\beta} = W(U_\psi, V_\psi) \neq 1.$$

Hence $T(X,Y) = 1$ admits a complete lift into generic point of $S(X) = 1$.

Case 2. A similar argument applies when $Comp(\Pi_{K(r+2)}) = 0$. Indeed, in this case for every $\sigma \in \mathcal{E}_{K(r+2)}$ the word $f_{M_{K(r+1)}}(\sigma)$ has length one, so $f_{M_{K(r+1)}}(\sigma) = \mu$ for some $\mu \in M_{K(r+2)}$. Now one can replace the word $V_\mu \in F[X \cup M_{\text{free}} \cup M_{\text{veryshort}}]$ by the label $f_{X_{K(r+1)}}(\sigma)$ where $f_{M_{K(r+1)}}(\sigma) = \mu$ and then repeat the argument.

$\square$

8. Implicit function theorem for NTQ systems

In this section we prove Theorems B, C, D from Introduction.

We begin with the proof of Theorem B. To this end let $U(X,A) = 1$ be a regular NTQ-system and $V(X,Y,A) = 1$ an equation compatible with $U = 1$. We need to show that $V(X,Y,A) = 1$ admits a complete effective U-lift.

We use induction on the number n of levels in the system $U = 1$. We construct a solution tree $T_{\mathrm{sol}}(V(X,Y,A) \wedge U(X,Y))$ with parameters $X = X_1 \cup \cdots \cup X_n$. In the terminal vertices of the tree there are generalized equations $\Omega_{v_1}, \ldots, \Omega_{v_k}$ which are equivalent to cut equations $\Pi_{v_1}, \ldots, \Pi_{v_k}$.

If $S_1(X_1, \ldots, X_n) = 1$ is an empty equation, we can take Merzljakov's words (see Introduction) as values of variables from X_1, express Y as functions in X_1 and a solution of some $W(Y_1, X_2, \ldots, X_n) = 1$ such that for any solution of the system

$$\begin{array}{rcl} S_2(X_2, \ldots, X_n, A) & = & 1 \\ & \vdots & \\ S_n(X_n, A) & = & 1 \end{array}$$

equation $W = 1$ has a solution.

Suppose, now that $S_1(X_1, \ldots, X_n) = 1$ is a regular quadratic equation. Let Γ be a basic sequence of automorphisms for the equation $S_1(X_1, \ldots, X_n, A) = 1$. Recall that

$$\phi_{j,p} = \gamma_j^{p_j} \cdots \gamma_1^{p_1} = \overleftarrow{\Gamma}_j^{p},$$

where $j \in \mathbb{N}$, $\Gamma_j = (\gamma_1, \ldots, \gamma_j)$ is the initial subsequence of length j of the sequence $\Gamma^{(\infty)}$, and $p = (p_1, \ldots, p_j) \in \mathbb{N}^j$. Denote by $\psi_{j,p}$ the following solution of $S_1(X_1) = 1$:

$$\psi_{j,p} = \phi_{j,p}\alpha,$$

where α is a composition of a solution of $S_1 = 1$ in G_2 and a solution from a generic family of solutions of the system

$$\begin{array}{rcl} S_2(X_2, \ldots, X_n, A) & = & 1 \\ & \vdots & \\ S_n(X_n, A) & = & 1 \end{array}$$

in $F(A)$. We can always suppose that α satisfies a small cancellation condition with respect to Γ.

Set

$$\Phi = \left\{ \phi_{j,p} \mid j \in \mathbb{N}, p \in \mathbb{N}^j \right\}$$

and let $\mathcal{L}^\alpha$ be an infinite subset of Φ^α satisfying one of the cut equations above. Without loss of generality we can suppose it satisfies Π_1. By Proposition 7.8 we can express variables from Y as functions of the set of Γ-words in X_1, coefficients, variables M_{free} and variables $M_{\mathrm{veryshort}}$, satisfying the system of equations $\Delta(M_{\mathrm{veryshort}})$ The system $\Delta(M_{\mathrm{veryshort}})$ can be turned into a generalized equation with parameters $X_2 \cup \cdots \cup X_n$, such that for any solution of the system

$$\begin{array}{rcl} S_2(X_2, \ldots, X_n, A) & = & 1 \\ & \vdots & \\ S_n(X_n, A) & = & 1 \end{array}$$

the system $\Delta(M_{\text{veryshort}})$ has a solution. Therefore, by induction, variables $(M_{\text{veryshort}})$ can be found as elements of G_2, and variables Y as elements of G_1. Theorem B is proved. $\square$

In order to prove Theorem C we need some auxiliary results.

LEMMA 8.1. *All stabilizing automorphisms (see [**9**]) of the left side of the equation*

$$c_1^{z_1}c_2^{z_2}(c_1c_2)^{-1} = 1 \tag{46}$$

have the form $z_1^{\phi} = c_1^k z_1(c_1^{z_1}c_2^{z_2})^n, z_2^{\phi} = c_2^m z_2(c_1^{z_1}c_2^{z_2})^n$. *All stabilizing automorphisms of the left side of the equation*

$$x^2c^z(a^2c)^{-1} = 1 \tag{47}$$

have the form $x^{\phi} = x^{(x^2c^z)^n}, z^{\phi} = c^k z(x^2c^z)^n$. *All stabilizing automorphisms of the left side of the equation*

$$x_1^2x_2^2(a_1^2a_2^2)^{-1} = 1 \tag{48}$$

have the form $x_1^{\phi} = (x_1(x_1x_2)^m)^{(x_1^2x_2^2)^n}$, $x_2^{\phi} = ((x_1x_2)^{-m}x_2)^{(x_1^2x_2^2)^n}$.

PROOF. The computation of the automorphisms can be done by software "Magnus". The statement of the lemma also follows from the fact that punctured surfaces corresponding to QH subgroups corresponding to these equations (see [**16**], Section 5) do not contain two intersecting simple closed curves that are not boundary-parallel. Therefore if G is a freely indecomposable finitely generated fully residually free group that has a QH subgroup Q corresponding to one of these equations, then G does not have two intersecting cyclic splittings with edge groups conjugated into Q. $\square$

If a quadratic equation $S(X) = 1$ has only commutative solutions then the radical $R(S)$ of $S(X)$ can be described (up to a linear change of variables) as follows (see [**12**]):

$$\text{Rad}(S) = ncl\{[x_i, x_j], [x_i, b], |\ \ i, j = 1, \ldots, k\},$$

where b is an element (perhaps, trivial) from F. Observe, that if b is not trivial then b is not a proper power in F. This shows that $S(X) = 1$ is equivalent to the system

$$U_{\text{com}}(X) = \{[x_i, x_j] = 1, [x_i, b] = 1, |\ \ i, j = 1, \ldots, k\}. \tag{49}$$

The system $U_{\text{com}}(X) = 1$ is equivalent to a single equation, which we also denote by $U_{\text{com}}(X) = 1$. The coordinate group $H = F_{R(U_{\text{com}})}$ of the system $U_{\text{com}} = 1$, as well as of the corresponding equation, is F-isomorphic to the free extension of the centralizer $C_F(b)$ of rank n. We need the following notation to deal with H. For a set X and $b \in F$ by $A(X)$ and $A(X, b)$ we denote free abelian groups with basis X and $X \cup \{b\}$, correspondingly. Now, $H \simeq F *_{b=b} A(X, b)$. In particular, in the case when $b = 1$ we have $H = F * A(X)$.

LEMMA 8.2. *Let* $F = F(A)$ *be a non-abelian free group and* $V(X, Y, A) = 1$, $W(X, Y, A) = 1$ *be equations over* F. *If a formula*

$$\Phi = \forall X(U_{\text{com}}(X) = 1 \to \exists Y(V(X, Y, A) = 1 \wedge W(X, Y, A) \neq 1))$$

is true in F *then there exists a finite number of extensions* ϕ_k *on* H *of* $\langle b\rangle$-*embeddings* $A(X, b) \to A(X, b)$ $(k \in K)$ *such that:*

(1) *every formula*

$$\Phi_k = \exists Y (V(X^{\phi_k}, Y, A) = 1 \wedge W(X^{\phi_k}, Y, A) \neq 1)$$

holds in the coordinate group $H = F *_{b=b} A(X, b)$*;*

(2) *for any solution* $\lambda : H \to F$ *there exists a solution* $\lambda^* : H \to F$ *such that* $\lambda = \phi_k \lambda^*$ *for some* $k \in K$*.*

Proof. We construct a set of initial parameterized generalized equations

$$\mathcal{G}E(S) = \{\Omega_1, \ldots, \Omega_r\}$$

for $V(X, Y, A) = 1$ with respect to the set of parameters X. For each $\Omega \in \mathcal{G}E(S)$, in [**16**, Section 8], we constructed the finite tree $T_{\rm sol}(\Omega)$ with respect to parameters X. Observe, that non-active part $[j_{v_0}, \rho_{v_0}]$ in the root equation $\Omega\Omega_{v_0}$ of the tree $T_{\rm sol}(\Omega)$ is partitioned into a disjoint union of closed sections corresponding to X-bases and constant bases (this follows from the construction of the initial equations in the set $\mathcal{G}E(S)$). We label every closed section σ corresponding to a variable $x \in X^{\pm 1}$ by x, and every constant section corresponding to a constant a by a. Due to our construction of the tree $T_{\rm sol}(\Omega)$ moving along a brunch B from the initial vertex v_0 to a terminal vertex v we transfer all the bases from the non-parametric part into parametric part until, eventually, in Ω_v the whole interval consists of the parametric part. For a terminal vertex v in $T_{\rm sol}(\Omega)$ equation Ω_v is periodized (see Section 5.4). We can consider the correspondent periodic structure $\mathcal{P}$ and the subgroup $\tilde{Z}_2$. Denote the cycles generating this subgroup by $z_1, \ldots, z_m$. Let $x_i = b^{k_i}$ and $z_i = b^{s_i}$. All x_i's are cycles, therefore the corresponding system of equations can be written as a system of linear equations with integer coefficients in variables $\{k_1, \ldots, k_n\}$ and variables $\{s_1, \ldots, s_m\}$:

$$k_i = \sum_{j=1}^{m} \alpha_{ij} s_j + \beta_i, \;\; i = 1, \ldots, n. \tag{50}$$

We can always suppose $m \leqslant n$ and at least for one equation Ω_v $m = n$, because otherwise the solution set of the irreducible system $U_{\rm com} = 1$ would be represented as a union of a finite number of proper subvarieties.

We will show now that all the tuples $(k_1, \ldots, k_n)$ that correspond to some system (50) with $m < n$ (the dimension of the subgroup H_v generated by $\bar{k} - \bar{\beta} = k_1 - \beta_1, \ldots, k_n - \beta_n$ in this case is less than n), appear also in the union of systems (50) with $m = n$. Such systems have form $\bar{k} - \bar{\beta}_q \in H_q$, q runs through some finite set Q, and where H_q is a subgroup of finite index in $Z^n = \langle s_1 \rangle \times \cdots \times \langle s_n \rangle$. We use induction on n. If for some terminal vertex v, the system (50) has $m < n$, we can suppose without loss of generality that the set of tuples H satisfying this system is defined by the equations $k_r = \ldots, k_n = 0$. Consider just the case $k_n = 0$. We will show that all the tuples $\bar{k}_0 = (k_1, \ldots, k_{n-1}, 0)$ appear in the systems (50) constructed for the other terminal vertices with $n = m$. First, if N_q is the index of the subgroup H_q, $N_q \bar{k} \in H_q$ for each tuple $\bar{k}$. Let N be the least common multiple of $N_1, \ldots, N_Q$. If a tuple $(k_1, \ldots, k_{n-1}, tN)$ for some t belongs to $\bar{\beta}_q + H_q$ for some q, then $(k_1, \ldots, k_{n-1}, 0) \in \bar{\beta}_q + H_q$, because $(0, \ldots, 0, tN) \in H_q$. Consider the set K of all tuples $(k_1, \ldots, k_{n-1}, 0)$ such that $(k_1, \ldots, k_{n-1}, tN) \notin \bar{\beta}_q + H_q$ for any $q = 1, \ldots, Q$ and $t \in \mathbb{Z}$. The set $\{(k_1, \ldots, k_{n-1}, tN) \mid (k_1, \ldots, k_{n-1}, 0) \in K, t \in \mathbb{Z}\}$ cannot be a discriminating set for $U_{\rm comm} = 1$. Therefore it satisfies some proper equation. Changing variables $k_1, \ldots, k_{n-1}$ we can suppose that for an irreducible

component the equation has form $k_{n-1} = 0$. The contradiction arises from the fact that we cannot obtain a discriminating set for $U_{\text{comm}} = 1$ which does not belong to $\bar{\beta}_q + H_q$ for any $q = 1, \ldots, Q$.

Embeddings ϕ_k are given by the systems (50) with $n = m$ for generalized equations Ω_v for all terminal vertices v. □

Below we describe two useful constructions. The first one is a *normalization* construction which allows one to rewrite effectively an NTQ-system $U(X) = 1$ into a normalized NTQ-system $U^* = 1$. Suppose we have an NTQ-system $U(X) = 1$ together with a fundamental sequence of solutions which we denote $\bar{V}(U)$.

Starting from the bottom we replace each non-regular quadratic equation $S_i = 1$ which has a non-commutative solution by a system of equations effectively constructed as follows.

1) If $S_i = 1$ is in the form

$$c_1^{x_{i1}} c_2^{x_{i2}} = c_1 c_2,$$

where $[c_1, c_2] \neq 1$, then we replace it by a system

$$\{\, x_{i1} = z_1 c_1 z_3,\ x_{i2} = z_2 c_2 z_3,\ [z_1, c_1] = 1,\ [z_2, c_2] = 1,\ [z_3, c_1 c_2] = 1 \,\}.$$

2) If $S_i = 1$ is in the form

$$x_{i1}^2 c^{x_{i2}} = a^2 c,$$

where $[a, c] \neq 1$, we replace it by a system

$$\{x_{i1} = a^{z_1}, x_{i2} = z_2 c z_1, [z_2, c] = 1, [z_1, a^2 c] = 1\}.$$

3) If $S_i = 1$ is in the form

$$x_{i1}^2 x_{i2}^2 = a_1^2 a_2^2$$

then we replace it by the system

$$\{\, x_{i1} = (a_1 z_1)^{z_2},\ x_{i2} = (z_1^{-1} a_2)^{z_2},\ [z_1, a_1 a_2] = 1,\ [z_2, a_1^2 a_2^2] = 1 \,\}.$$

The normalization construction effectively provides an NTQ-system $U^* = 1$ such that each solution in $\bar{V}(U)$ can be obtained from a solution of $U^* = 1$. We refer to this system as to the normalized system of U corresponding to $\bar{V}(U)$. Similarly, the coordinate group of the normalized system is called the *normalized* coordinate group of $U = 1$.

Lemma 8.3. *Let $U(X) = 1$ be an NTQ-system, and $U^* = 1$ be the normalized system corresponding to the fundamental sequence $\bar{V}(U)$. Then the following holds:*

(1) *The coordinate group $F_{R(U)}$ canonically embeds into $F_{R(U^*)}$;*
(2) *The system $U^* = 1$ is an NTQ-system of the type*

$$\begin{aligned} S_1(X_1, X_2, \ldots, X_n, A) &= 1 \\ S_2(X_2, \ldots, X_n, A) &= 1 \\ &\vdots \\ S_n(X_n, A) &= 1 \end{aligned}$$

in which every $S_i = 1$ is either a regular quadratic equation or an empty equation or a system of the type

$$U_{\text{com}}(X, b)\{[x_i, x_j] = 1, [x_i, b] = 1 \mid i, j = 1, \ldots, k\}$$

where $b \in G_{i+1}$.

(3) *Every solution* X_0 *of* $U(X) = 1$ *that belongs to the fundamental sequence* $\bar{V}(U)$ *can be obtained from a solution of the system* $U^* = 1$.

PROOF. Statement (1) follows from the normal forms of elements in free constructions or from the fact that applying standard automorphisms ϕ_L to a non-commuting solution (in particular, to a basic one) one obtains a discriminating set of solutions (see Section 7.2). Statements (2) and (3) are obvious from the normalization construction. □

DEFINITION 8.4. A family of solutions Ψ of a regular NTQ-system $U(X, A) = 1$ is called *generic* if for any equation $V(X, Y, A) = 1$ the following is true: if for any solution from Ψ there exists a solution of $V(X^\psi, Y, A) = 1$, then $V = 1$ admits a complete U-lift.

A family of solutions Θ of a regular quadratic equation $S(X) = 1$ over a group G is called *generic* if for any equation $V(X, Y, A) = 1$ with coefficients in G the following is true: if for any solution $\theta \in \Theta$ there exists a solution of $V(X^\theta, Y, A) = 1$ in G, then $V = 1$ admits a complete S-lift.

A family of solutions Ψ of an NTQ-system $U(X, A) = 1$ is called *generic* if $\Psi = \Psi_1 \ldots \Psi_n$, where Ψ_i is a generic family of solutions of $S_i = 1$ over G_{i+1} if $S_i = 1$ is a regular quadratic system, and Ψ_i is a discriminating family for $S_i = 1$ if it is a system of the type U_{com}.

The second construction is a *correcting extension of centralizers* of a normalized NTQ-system $U(X) = 1$ relative to an equation $W(X, Y, A) = 1$, where Y is a tuple of new variables. Let $U(X) = 1$ be an NTQ-system in the normalized form:

$$\begin{aligned} S_1(X_1, X_2, \ldots, X_n, A) &= 1 \\ S_2(X_2, \ldots, X_n, A) &= 1 \\ &\vdots \\ S_n(X_n, A) &= 1. \end{aligned}$$

So every $S_i = 1$ is either a regular quadratic equation or an empty equation or a system of the type

$$U_{\text{com}}(X, b) = \{[x_i, x_j] = 1, [x_i, b] = 1, \mid \ i, j = 1, \ldots, k\}$$

where $b \in G_{i+1}$. Again, starting from the bottom we find the first equation $S_i(X_i) = 1$ which is in the form $U_{\text{com}}(X) = 1$ and replace it with a new centralizer extending system $\bar{U}_{\text{com}}(X) = 1$ as follows.

We construct T_{sol} for the system $W(X, Y) = 1 \wedge U(X) = 1$ with parameters $X_i, \ldots, X_n$. We obtain generalized equations corresponding to final vertices. Each of them consists of a periodic structure on X_i and generalized equation on $X_{i+1} \ldots X_n$. We can suppose that for the periodic structure the set of cycles $C^{(2)}$ is empty. Some of the generalized equations have a solution over the extension of the group G_i. This extension is given by the relations $\bar{U}_{\text{com}}(X_i) = 1, S_{i+1}(X_{i+1}, \ldots, X_n) = 1, \ldots, S_n(X_n) = 1$, so that there is an embedding $\phi_k : A(X, b) \to A(X, b)$. The others provide a proper (abelian) equation $E_j(X_i) = 1$ on X_i. The argument above shows that replacing each centralizer extending system $S_i(X_i) = 1$ which is in the form $U_{\text{com}}(X_i) = 1$ by a new system of the type $\bar{U}_{\text{com}}(X_i) = 1$ we eventually rewrite the system $U(X) = 1$ into finitely many new

ones $\bar{U}_1(X) = 1, \ldots, \bar{U}_m(X) = 1$. We denote this set of NTQ-systems by $\mathcal{C}_W(U)$. For every NTQ-system $\bar{U}_m(X) = 1 \in \mathcal{C}_W(U)$ the embeddings ϕ_k described above give rise to embeddings $\bar{\phi} : F_{R(U)} \to F_{R(\bar{U})}$. Finally, combining normalization and correcting extension of centralizers (relative to $W = 1$) starting with an NTQ-system $U = 1$ and a fundamental sequence of its solutions $\bar{V}(U)$ we can obtain a finite set

$$\mathcal{N}C_W(U) = \mathcal{C}_W(U^*)$$

which comes equipped with a finite set of embeddings $\theta_i : F_{R(U)} \to F_{R(\bar{U}_i)}$ for each $\bar{U}_i \in \mathcal{N}C_W(U)$. These embeddings are called *correcting normalizing embeddings.* The construction implies the following result.

THEOREM 8.5. *Let $U(X, A) = 1$ be an NTQ-system with a fundamental sequence of solutions $V_{\mathrm{fund}}(U)$. If a formula*

$$\Phi = \forall X(U(X) = 1 \to \exists Y(W(X, Y, A) = 1 \wedge W_1(X, Y, A) \neq 1)$$

is true in F. Then for every $\bar{U}_i \in \mathcal{N}C_W(U)$ the formula

$$\exists Y(W(X^{\theta_i}, Y, A) = 1 \wedge W_1(X^{\theta_i}, Y, A) \neq 1)$$

is true in the group $F_{R(\bar{U}_i)}$ for every correcting normalizing embedding

$$\theta_i : F_{R(U)} \to F_{R(\bar{U}_i)}.$$

Furthermore, for every fundamental solution $\phi : F_{R(U)} \to F$ there exists a fundamental solution ψ of one of the systems $\bar{U}_i = 1$, where $\bar{U}_i \in \mathcal{N}C_W(U)$ such that $\phi = \theta_i\psi$.

COROLLARY 8.6. *Theorem C holds.*

Now we are ready to prove Theorem D.

PROOF OF THEOREM D. By [**16**, Theorem 11.1] for a finite system of equations $U = 1$ over F one can effectively find NTQ systems $U_i = 1$, $i = 1, \ldots, k$ and homomorphisms $\theta_i : F_{R(U)} \to F_{R(U_i)}$ such that for every solution ϕ of $U = 1$ there exists i such that $\phi = \theta_i\psi$, where $\psi \in V_{\mathrm{fund}}(U_i)$. Now the result follows from Theorem C. $\square$

9. Groups that are elementary equivalent to a free group

In this section we prove Theorem E from the introduction.

Let $\mathcal{C}$ ($\mathcal{C}^*$) be the class of finite systems $U(X) = 1$ over F such that every equation $T(X, Y) = 1$ compatible with $U(X) = 1$ admits U-lift (complete U-lift). We showed in Section 2, Lemma 2.9, that these classes are closed under rational equivalence. Denote by $\mathcal{K}$ the class of the coordinate groups $F_{R(U)}$ of systems $U(X) = 1$ over F such that every equation $T(X, Y) = 1$ over F compatible with $U(X) = 1$ admits a U-lift. It follows that every finite set of defining relations of a group from $\mathcal{K}$ gives rise to a system from $\mathcal{C}$.

By Theorem B the class $\mathcal{K}$ contains the coordinate groups of regular NTQ systems.

Below, in the case of a coefficient-free system $S(X) = 1$ we put $G_{cfR(S)} = F(X)/R(S)$, then $G_{R(S)} = G * G_{cfR(S)}$. In this case the group $G_{cfR(S)}$ can be also viewed as the coordinate group of $V(S)$. It is usually clear from the context which groups is considered in the case of the coefficient-free system.

Lemma 9.1. *The class $\mathcal{K}$ is closed under retracts. Namely, if H is a finitely generated subgroup of G such that there exists a retract $\phi : G \to H$. Then:*

(1) *if $F \leqslant H$ then $H = F_{R(U)}$ for some system $U = 1$ over F and every equation compatible with $U = 1$ admits a U-lift;*
(2) *if $F \cap H = 1$ then $H = F_{R(U)}$ for some coefficient-free system $U = 1$ over F and every coefficient-free equation compatible with $U = 1$ admits a U-lift into $F_{cf\,R(S)}$.*

Proof. We show only (1), but a similar argument proves (2). Let $H = \langle F \cup X_1 \rangle$ be a finitely generated subgroup of G generated by F and a finite set X_1. Then H is residually free, so $H = F_{R(U)}$ for some system $U(X_1) = 1$ over F. Since H is a subgroup of G it follows that $X_1 = P(X)$ for some word mapping P. If $T(X_1, Y) = 1$ is compatible with $U(X_1) = 1$ then $T(P(X), Y) = 1$ is compatible with $S(X) = 1$. Therefore $T(P(X), Y) = 1$ admits an S-lift, so $T(P(X), V(X)) = 1$ in G for some $V(X) \in G$. It follows that

$$T(P(X), V(X))^{\phi} = T(P(X)^{\phi}, V(X^{\phi})) = T(P(X), V(X^{\phi})) = T(X_1, V(X^{\phi})) = 1$$

so $T(X_1, Y) = 1$ admits a U-lift. $\square$

Corollary 9.2. *The class $\mathcal{K}$ is closed under free factors. Namely, if $G \in \mathcal{K}$ then every factor in a free decomposition of G modulo F belongs to $\mathcal{K}$.*

Theorem E. *Let F be a free non-abelian group and $S(X) = 1$ a consistent system of equations over F. Then the following conditions are equivalent:*

(1) *The system $S(X) = 1$ is rationally equivalent to a regular NTQ system.*
(2) *Every equation $T(X, Y) = 1$ which is compatible with $S(X) = 1$ over F admits an S-lift.*
(3) *Every equation $T(X, Y) = 1$ which is compatible with $S(X) = 1$ over F admits a complete S-lift.*

Proof. (1) $\Longrightarrow$ (3). It follows from Lemma 2.9 which states that the class $\mathcal{C}^*$ is closed under rational equivalence and the fact that $\mathcal{C}^*$ contains all regular NTQ systems (Theorem B).

(3) $\Longrightarrow$ (2). Obvious.

(2) $\Longrightarrow$ (1). Suppose that every equation which is compatible with $S = 1$ over F admits an S-lift. Consider $G = F_{R(S)}$.

Lemma 9.3. *The group G does not have non-cyclic abelian subgroups.*

Proof. Suppose G has a non-cyclic abelian subgroup, let x, y be two basis elements in this subgroup. Consider their expressions in generators of G: $x = u(X)$, $y = v(X)$. Then the system of equations

$$S_1(X, x, y) = (S(X) = 1 \wedge x = u(X) \wedge y = v(X) \wedge [x, y] = 1)$$

is rationally equivalent to $S(X) = 1$, therefore every system of equations compatible with $S_1(X, x, y) = 1$ admits an S-lift. The formula

$$\forall X \forall x \forall y \exists u (S_1(X, x, y) = 1 \to (u^2 = x \vee u^2 = y \vee u^2 = xy))$$

is true in every free group, because in a free group the images of x, y are powers of the same element. But this formula is false in G. Therefore the system

$$u^2 = x \vee u^2 = y \vee u^2 = xy$$

does not admit an S-lift. This gives a contradiction to the assumption. $\square$

By Corollary 9.2 we may assume that G is freely indecomposable. There are two cases to consider, $F \leqslant G$ and $F \cap G = 1$. Since the same argument gives a proof for both of them we consider only one case, say $F \leqslant G$.

If G does not have a non-degenerate JSJ $\mathbb{Z}$-decomposition [**16**] then G is either a surface group, or G is an infinite cyclic group (in the case $F \cap G = 1$). In both cases G is the coordinate group of a regular NTQ system, as required.

Suppose now, that G has a non-degenerate JSJ $\mathbb{Z}$-decomposition of G, say D. Denote by $\langle X \mid U \rangle$ the canonical finite presentation of G as the fundamental group of the graph of groups D. By Lemma 2.9 the class $\mathcal{C}$, of systems $V = 1$ over F for which every compatible equation admits an V-lift, is closed under rational equivalence. Hence $U = 1$ belongs to $\mathcal{C}$. Since $G = F_{R(U)}$ we may assume from the beginning that $S = U$, so $G = \langle X \mid S \rangle$ is the canonical finite presentation of G as the fundamental group of D.

Let A_E be the group of automorphisms (F-automorphisms, in the case $F \leqslant G$) of G generated by Dehn's twists along the edges of D. The group A_E is abelian by Lemma 2.25 [**16**]. Recall, that two solutions ϕ_1 and ϕ_2 of the equation $R(X) = 1$ are *A_E-equivalent* if there is an automorphism $\sigma \in A_E$ such that $\sigma\phi_1 = \phi_2$.

Recall, that if A is a group of canonical automorphisms of G then the the maximal standard quotient of G with respect to A is the quotient G/R_A of G by the intersection R_A of the kernels of all solutions of $S(X) = 1$ which are minimal with respect to A (see [**16**] for details).

By [**16**, Theorem 9.1] the maximal standard quotient G/R_{A_D} of with respect to the whole group of canonical automorphisms A_D is a proper quotient of G, i.e., there exists an equation $V(X) = 1$ such that $V \notin R(S)$ and all minimal solution of $S(X) = 1$ with respect to the canonical group of automorphisms A_D satisfy the equation $V(X) = 1$. Now, compare this with the following result.

LEMMA 9.4. *The maximal standard quotient of G with respect to the group A_E is equal to G, i.e., the set of of minimal solutions with respect to A_E discriminates G.*

PROOF. Suppose, to the contrary, that the standard minimal quotient G/R_{A_E} of G is a proper quotient of G, i.e., there exists $V \in G$ such that $V \neq 1$ and $V^\phi = 1$ for any minimal solution of S with respect to A_E. Recall that the group A_E is generated by Dehn twists along the edges of D. If c_e is a given generator of the cyclic subgroup associated with the edge e, then we know how the Dehn twists $\sigma = \sigma_e$ associated with e acts on the generators from the set X. Namely, if $x \in X$ is a generator of a vertex group, then either $x^\sigma = x$ or $x^\sigma = c^{-1}xc$. Similarly, if $x \in X$ is a stable letter then either $x^\sigma = x$ or $x^\sigma = xc$. It follows that for $x \in X$ one has $x^{\sigma^n} = x$ or $x^{\sigma^n} = c^{-n}xc^n$ $[x^{\sigma^n} = xc^n]$ for every $n \in \mathbb{Z}$. Now, since the centralizer of c_e in G is cyclic (Lemma 9.3) the following equivalence holds:

$$\exists n \in \mathbb{Z}(x^{\sigma^n} = z) \Longleftrightarrow \begin{cases} \exists y([y, c_e] = 1 \wedge y^{-1}xy = z) & \text{if } x^\sigma = c_e^{-1}xc_e; \\ x = z & \text{if } x^\sigma = x. \end{cases}$$

Similarly, since the group A_E is finitely generated abelian one can write down a formula which describes the relation

$$\exists \alpha \in A_e(x^\alpha = z)$$

One can write the elements c_e as words in generators X, say $c_e = c_e(X)$. Now the formula

$$\forall X \exists Y \exists Z \left(S(X) = 1 \to \left(\bigwedge_{i=1}^{m} [y_i, c_i(X)] = 1 \wedge Z = X^{\sigma_Y} \wedge V(Z) = 1 \right) \right)$$

holds in the group F. Indeed, this formula tells one that each solution of $S(X) = 1$ is A_E-equivalent to (a minimal) solution that satisfies the equation $V(X) = 1$. Since $S(X) = 1$ is in $\mathcal{C}$ the system

$$\left(\bigwedge_{i=1}^{m} [y_i, c_i(X)] = 1 \wedge Z = X^{\sigma_Y} \wedge V(Z) = 1 \right)$$

admits an S-lift. Hence there is an automorphism $\alpha \in A_E$ such that $V(X^\alpha) = 1$ in G, so $V(X) = 1$ – contradiction. □

Lemma 9.5. *There exist QH subgroups in D.*

Proof. By Theorem 9.1 [**16**] the maximal standard quotient G/R_{A_D} of G with respect to the whole group A_D of the standard automorphisms of G is a proper quotient of G. Let E_1 be the set of edges between non-QH vertex groups. By [**16**, Lemma 2.25] the group A_D is a direct product of A_{E_1} and the group generated by the canonical automorphisms corresponding to QH vertices and abelian non-cyclic vertex groups. By Lemma 9.3 there are no abelian non-cyclic groups in D, so A_D is a direct product of A_{E_1} and the group generated by the canonical automorphisms of QH vertices. Since the maximal standard quotient of G with respect to A_E is not proper (Lemma 9.4) then $A_D \neq A_E$ hence (see Section 2.20 in [**16**]) D has QH subgroups. □

Let $K = \langle X_2 \rangle$ be the fundamental group of the graph of groups obtained from D by removing all QH subgroups.

Lemma 9.6. *The natural homomorphism $G \to G/R_D$ is a monomorphism on K.*

Proof. This follows from Lemma 9.4 and the fact that canonical automorphisms corresponding to QH subgroups fix K. □

Lemma 9.7. *There is a K-homomorphism ϕ from G into itself with the non-trivial kernel.*

Proof. The generating set X of G corresponding to the decomposition D can be partition as $X = X_1 \cup X_2$. Consider a formula

$$\begin{aligned} &\forall X_1 \forall X_2 \exists Y \exists T \exists Z \, (S(X_1, X_2) = 1 \\ &\qquad \to \left(\bigwedge_{i1}^{m} [t_i, c_i(X_2)] = 1 \wedge Z = X_2^{\sigma_T} \wedge S(Y, X_2) = 1 \wedge V(Y, Z) = 1 \right) \Bigg). \end{aligned}$$

It says that each solution of the equation $S(X_1, X_2) = 1$ can be transformed by a canonical automorphism into a solution Y, Z that satisfies $V(Y, Z) = 1$. It is true in a free group, therefore the system

$$\left(\bigwedge_{i=1}^{m} [t_i, c_i(X_2)] = 1 \wedge Z = X_2^{\sigma_T} \wedge S(Y, X_2) = 1 \wedge V(Y, Z) = 1 \right)$$

can be lifted in G. Elements Z generate the same subgroup K as X_2, because $t_i = c_i^{n_i}$, for a fixed number n_i, $i = 1, \dots, m$ in G. Therefore, there is a proper K-homomorphism ϕ from G into itself. □

For a QH subgroup Q we denote by P_Q the fundamental group of the graph of groups obtained from D by removing the QH-vertex v_Q and all the adjacent edges. In the following lemma, the second statement in not needed for the proof of Theorem E, but we included it for completeness.

Lemma 9.8.

1. *There exists a QH subgroup Q in D such that P_Q is a retract.*
2. *The maximal standard quotient G/R_{A_Q} of G, with respect to the group A_Q of the canonical automorphisms of G corresponding to Q, is a proper quotient of G.*

Proof. 1. The image $H = \phi(G)$ cannot contain conjugates of finite index subgroups of all the QH subgroups of D. Indeed, suppose it does. Let $Q_1, \dots, Q_s$ be QH subgroups with minimal number of free generators. There is no homomorphism from a finitely generated free group onto a proper finite index subgroup. Therefore the family $Q_1, \dots, Q_s$ has to be mapped onto the same family of QH subgroups. Similarly, the family of all QH subgroups would be mapped onto the conjugates of subgroups from the same family, and different QH subgroups would be mapped onto conjugates of different QH subgroups. In this case H would be isomorphic to G. This is impossible because G is hopfian. Therefore there is a QH subgroup Q such that H does not intersect any conjugate Q^g in a subgroup of finite index.

By construction, G is the fundamental group of the graph of groups with vertex v_Q and vertices corresponding to connected components $Y_1, \dots, Y_k$ of the graph for P_Q. Let $P_1, \dots, P_k$ be the fundamental groups of the graph of groups on $Y_1, \dots, Y_k$. Then $P_Q = P_1 * \cdots * P_k$. Let D_Q be a JSJ decomposition of G modulo K. Then it has two vertices v_Q and the vertex with vertex group P_Q.

By [**16**, Lemma 2.13] applied to D_Q and the subgroup H, one of the following holds:

(1) H is a nontrivial free product modulo K;
(2) $H \leqslant P_Q^g$ for some $g \in G$.

Moreover, the second statement of this lemma is the following. If $H_Q = H \cap Q$ is non-trivial and has infinite index in Q, then H_Q is a free product of some conjugates of $p_1^{\alpha_1}, \dots, p_m^{\alpha_m}, p^{\alpha}$ and a free group F_1 (maybe trivial) which does not intersect any conjugate of $\langle p_i \rangle$ for $i = 1, \dots, m$.

In the case (2) one has $H \leqslant P_Q^g$, and, conjugating, we can suppose that $H \leqslant P_Q$.

Suppose now that the case (1) holds. For any g the subgroup $Q^g \cap H$ is either trivial or has the structure described in the second statement of Lemma 2.13, [**16**]. Consider now the decomposition D_H induced on H from D_Q. If the group F_1 is nontrivial, then H is freely decomposable modulo K, because the vertex group Q_H in D_H is a free product, and all the edge groups belong to the other factor. If at least for one subgroup Q^g, such a group F_1 is non-trivial, then H is a non-trivial free product and the subgroup K belongs to the other factor. Hence $H = H_1 * T$, where $K \in H_1$. In this case we consider $\phi_1 = \phi\psi$, where ψ is identical on H_1 and $\psi(x) = 1$ for $x \in T$. Now each non-trivial subgroup $H_1 \cap Q^g$ is a free product of conjugates of some elements $p_i^{\alpha_i}$, $\alpha_i \in Z$, in Q^g.

According to the Bass-Serre theory, for the group G and its decomposition D_Q one can construct a tree such that G acts on this tree, and stabilizers correspond to vertex and edge groups of D_Q. Denote this Bass-Serre tree by T_{D_Q}. The subgroup H_1 also acts on T_{D_Q}. Let T_1 be a fundamental transversal for this action. Either $H_1 \leqslant P_Q^g$ or H_1 is not conjugated into P_Q. The amalgamated product of the stabilizers of the vertices of T_1 is a free product of subgroups $H_1 \cap P_Q^g$. Therefore H_1 is either such a free product or is obtained from such a free product by a sequence of HNN extensions with associated subgroups belonging to distinct factors of the free product. In both cases H_1 is freely decomposable modulo K. Conjugating, we can suppose that one of the factors of $\phi_1(G)$ is contained in P_Q. We replace now ϕ_1 by ϕ_2 which is a composition of ϕ_1 with the homomorphism identical on the factor that is contained in P_Q and sending the other free factors into the identity. Then $\phi_2(G) = H_2 \leqslant P_Q$, where H_2 is freely indecomposable modulo K.

A mapping π defined on the generators X of G as

$$\pi(x) = \begin{cases} \phi_2(x) & \texttt{if } x \in Q; \\ x & \texttt{if } x \notin Q \end{cases}$$

can be extended to a proper homomorphism π from G onto P_Q. Then π is a P_Q-homomorphism, and P_Q is a retract.

2. Let $X = X_3 \cup X_4$ be a partition of X such that X_4 are generators of P_Q. Then the following formula is true in G

$$\forall X_3 \forall X_4 \exists Y (S(X_3, X_4) = 1 \to (S(Y, X_4) = 1 \wedge Y = r(X_4))),$$

where $Y = r(X_4) = \pi(X_3)$. This formula is also true in F.

For a homomorphism $\gamma : G \to F$ there are two possibilities:

a) γ can be transformed by a canonical automorphism from A_Q into a homomorphism $\beta : G \to F$, such that there exists $\alpha : G \to P_Q * F(Z)$ and $\psi : P_Q * F(Z) \to F$ such that $\beta = \alpha\psi$. Here $F(Z)$ is a free group corresponding to free variables of the quadratic equation corresponding to Q.

b) γ is a solution of one of the finite number of proper equations that correspond to the cases $\gamma(Q)$ is abelian or $\gamma(G_e) = 1$, where e is an edge adjacent to v_Q.

Since $\ker(\alpha) = \bigcap \ker(\alpha\psi)$, where $\psi \in Hom(P_Q * F(Z), F)$, the statement follows. □

By Lemma 9.1 the group $P = P_Q$ belongs to $\mathcal{K}$. If P is freely undecomposable [modulo F] and does not have a non-degenerate JSJ decomposition [modulo F] then H is either F or a cyclic group, or a surface group. In this event, G is a regular NTQ (since only regular quadratic equations belong to the class $\mathcal{C}$). If P is freely decomposable modulo F or it has a non-degenerate JSJ decomposition we put $G_0 = G$, $Q_0 = Q$ and repeat the argument above to the group $G_1 = P$. Thus, by induction we construct a sequence of proper epimorphisms:

$$G \to G_1 \to G_2 \to \dots$$

and a sequence of QH subgroups Q_i of the groups G_i such that G_i is the fundamental group of the graph of groups with two vertices Q_i and G_{i+1} and such that Q_i is defined by a regular quadratic equation $S_i = 1$ over G_{i+1} and such that $S_i = 1$ has a solution in G_{i+1}. Since free groups are equationally Noetherian this sequence terminates in finitely many steps either at a surface group, or the free group F, or an infinite cyclic group. This shows that the group G is F-isomorphic to a coordinate group of some regular NTQ system.

This proves the theorem. □

As a corollary one can obtain the following result. To explain we need few definitions. Let F be a free group and L_F be a group theory language with constants from the group F, and Φ be a set of first order sentences of the language L_F. Recall, that two groups G and H are Φ-equivalent if they satisfy precisely the same sentences from the set Φ. In this event we write $G \equiv_{\Phi} H$. In particular, $G \equiv_{\forall\exists} H$ ($G \equiv_{\exists\forall} H$) means that G and H satisfy precisely the same $\forall\exists$-sentences ($exists\forall$-sentences). Notice that $G \equiv_{\forall\exists} H \iff G \equiv_{\exists\forall} H$. We have shown in [**13**] that for a finitely generated group G $G \equiv_{\forall\exists} H$ implies that G is torsion-free hyperbolic. Now we can prove Theorem F from the introduction:

Theorem F. *Every finitely generated group which is $\forall\exists$-equivalent to a free non-abelian group F is isomorphic to the coordinate group of a regular NTQ system over F.*

Proof of Theorem F. Let G be a finitely generated group which is $\forall\exists$-equivalent to a free non-abelian group F. In particular, G is $\forall$-equivalent to F, hence by Remeslennikov's theorem [**25**] the group G is fully residually free. It follows then that G is the coordinate group of some irreducible system $S = 1$ over F (see [**2**]), so $G = F_{R(S)}$. We claim that every equation compatible with $S(X) = 1$ admits an S-lift over F. Indeed, if $T(X, Y) = 1$ is compatible with $S(X) = 1$ over F then the formula

$$\forall X \exists Y (S(X) = 1 \rightarrow T(X, Y) = 1)$$

is true in F, hence in G. Therefore, the equation $T(X^{\mu}, Y) = 1$ has a solution in G for any specialization of variables from X in G, in particular, for the canonical generators X of G. This shows that every equation compatible with $S = 1$ admits S-lift. By Theorem E, the group G is isomorphic to the coordinate group of a regular NTQ system, as required. □

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